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MOISTURE MOVEMENT IN A HORIZONTAL SOIL COLUMN UNDER THE INFLUENCE OF AN APPLIED PRESSURE

WILLIAM WEN-GONG YEH

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16. Abstract The horizontal movement of moisture in unsaturated homogeneous soil is described by the diffusion equation for particular boundary conditions. The problem under consideration is that of horizontal moisture movement in a semi-infinite column of homogeneous soil initially at a uniform moisture content and with flow introduced under a certain applied constant pressure at one end. The soil-water system is subdivided into saturated and unsaturated zones. The liquid pore pressure in the saturated zone varies linearly with distance from the applied pressure at the soil face to the critical-suction value at the interface between the saturated and unsaturated zones. A theoretical solution was obtained by combining Darcy's equation with a modified diffusion equation that accounted for the moving interface between the two zones. The nonlinear partial differential eqution of dufission is transformed to a nonlinear ordinary differential equation and a numerical procedure for its solution is presented. Simplified flow equations assuming complete saturation to the wetting from and their solutions are presented and compared with the more rigorous approach.
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Abstractor T. H. Woodard 1/4/72

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STANFORD UNIVERSITY
STANFORD, CALIFORNIA

MOISTURE MOVEMENT IN A HORIZONTAL SOIL COLUMN UNDER THE INFLUENCE OF AN APPLIED PRESSURE

By

William Wen-Gong Yeh
Ph.D.

Technical Report No. 77
March 1967

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I

This report was prepared originally as a dissertation submitted to the Department of Civil Engineering and the Graduate Division of Stanford University in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

Recent studies have shown that the horizontal movement of moisture in unsaturated homogeneous soil can be described by the diffusion equation for particular boundary conditions,

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\ \frac{\partial\theta}{\partial x}\right] \quad ,$$

where θ is the moisture content (cm^3 per cm^3), t is time (min), and D is the diffusivity (cm^2 per min). The diffusivity embodies the hydraulic conductivity and capillary characteristics of the soil.

The problem under consideration is that of horizontal moisture movement in a semi-infinite column of homogeneous soil initially at a uniform moisture content and with flow introduced under a certain applied constant pressure at one end. The soil-water system is subdivided into two distinct zones, i.e., the saturated and unsaturated zones. The liquid pore pressure in the saturated zone varies linearly with distance from the applied pressure at the soil face to the critical-suction value at the interface between the saturated and unsaturated zones. A theoretical solution was obtained by combining Darcy's equation with a modified diffusion equation that accounted for the moving interface between the two zones. The non-linear partial differential equation of duffision is transformed to a non-linear ordinary differential equation and a numerical procedure for its solution is presented. Application of the theoretical results requires knowledge of the hydraulic characteristics and the diffusivity function of the soil. Simplified flow equations assuming complete saturation to the wetting front and their solutions are presented and compared with the more rigorous approach.

After experimentally determining the hydraulic characteristics and diffusivity function of the soil (silica flour), a set of experiments was conducted by admitting water to the soil sample under different applied hydrostatic pressures. The moisture content as a function of time and position was determined by use of gamma-ray attenuation equipment.

Statistical considerations in the use of gamma-radiation attenuation techniques are presented. The experimental results checked the analytical results quite closely thus confirming the validity of the analytical approach.

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CHAPTER 1

INTRODUCTION AND REVIEW OF PREVIOUS WORK

1.1 Background

The fundamentals of groundwater movement were established only a century ago; within recent years the subject has been treated more rigorously with the help of mathematics and high-speed computers. In 1856 Henri Darcy studied the flow of water through filter sands and discovered the law of non-turbulent flow of fluids through a porous medium; this has since been designated as "Darcy's law". Dupuit, Theis and others applied Darcy's law to describe flow towards a well under different conditions. Many boundary value problems such as seepage from dams, etc., have since been solved (Harr, 1962)[1].

The flow through a porous medium may be steady or unsteady, depending on whether or not the flow changes with time. Also the medium may be saturated or unsaturated. In unsaturated flow the pore spaces are not completely filled with flowing liquid. Typical examples of unsteady unsaturated flow are infiltration into the soil surface and capillary rise from a water table. The hydrologist is particularly interested in infiltration, for if infiltration rates are known, runoff rates may be computed provided that precipitation rates are known. The rate of intake plays a very important role in groundwater recharge and waste water disposal as well as in irrigation. In many such cases, the flow is unsteady and unsaturated. This problem is complicated by the presence of capillary forces. The diffusion equation describing such phenomena was first introduced by Childs and Collis-George (1948).

The solution of the nonlinear diffusion equation for horizontal flow has been studied by many investigators including Crank and Henry (1949), Klute (1952), Philip (1957 ab), Gardner and Mayhugh (1958), Kobayaski (1960), Scott (1962), Ashcroft et al. (1962), and Hanks and Bowers (1962). The Boltzmann transformation has been successfully used in the solution of this problem. By numerical methods utilizing high-speed computers the validity of the Boltzmann transformation has been verified. Kobayaski applied a complex-variable transformation to the nonlinear diffusion equation. The transformation allows the partial differential equation to be reduced to an ordinary differential equation which can be solved by direct integration. However, the transformation is only applicable for certain limited boundary conditions. Crank and Henry used the Boltzmann transformation to reduce the partial differential equation which in principle can be solved by standard iterative methods. However, when diffusivity varies markedly with moisture content, Crank and Henry's method is very inefficient and solutions near the initial moisture content are erroneous. Philip used the Boltzmann transformation and solved the transformed equation by a series approximation. The series solution fails when the time is very large and assumptions have to be made for such a case. Scott et al. solved the diffusion equation by power series expansion for exponential and linear diffusivity functions. Ashcroft et al. developed a numerical technique for solving the implicit difference analogues of the diffusion equation. The method involves (1) devising suitable difference analogues for the partial differential equation, (2) developing a method for choosing diffusivity to convert the nonlinear difference analogues into linear equations, and (3) solving the system of linear equations. The solutions obtained in this method were found to give good agreement with the solutions obtained using the Boltzmann transformation. Hanks and Bowers solved the one-dimensional equation for infiltration into layered soils. The numerical solution was found to give good results when compared with methods of Scott et al. and Philip for horizontal flow. Swartzendruber (1966) has shown that because of the product form of variables-separation that the Boltzmann transformation appears as a consequence of the governing partial differential equation and one boundary condition. Singh (1965) has obtained a solution to the two-dimensional diffusion from a cylindrical source of finite dimension. Philip (1958) using the diffusion approach analagous to heat conduction analyzed the case of infiltration with water ponded on the surface by a series approximation limited to a certain radius of convergence.

---

[1] References are arranged alphabetically in the bibliography.

Experiments by Nielson et al. (1962) have demonstrated the inability of the diffusion equation to predict water movement for large negative applied pressure. Laboratory experiments by Bodman and Colman (1944) on one-dimensional infiltration from a ponded water surface have shown that the moisture profile may be subdivided into two or more distinct zones. Below the soil surface there is a large layer of full saturation and the pressure in the liquid is greater than atmospheric pressure. This zone is referred to as the saturated zone, and its lower boundary is defined as the saturated wetting front. Below the saturated wetting front the soil moisture decreases rapidly with depth in what may be termed the unsaturated zone; the lower boundary of this zone is called the unsaturated wetting front. The pressure in the liquid in this region is less than atmospheric pressure.

1.2 The Problem Under Consideration

The problem investigated in this study is that of moisture movement in a horizontal semi-infinite homogeneous soil column initially at a uniform moisture content under the influence of a certain applied pressure at one end. A schematic picture of this problem is depicted in Figs. 1.1 and 1.2.

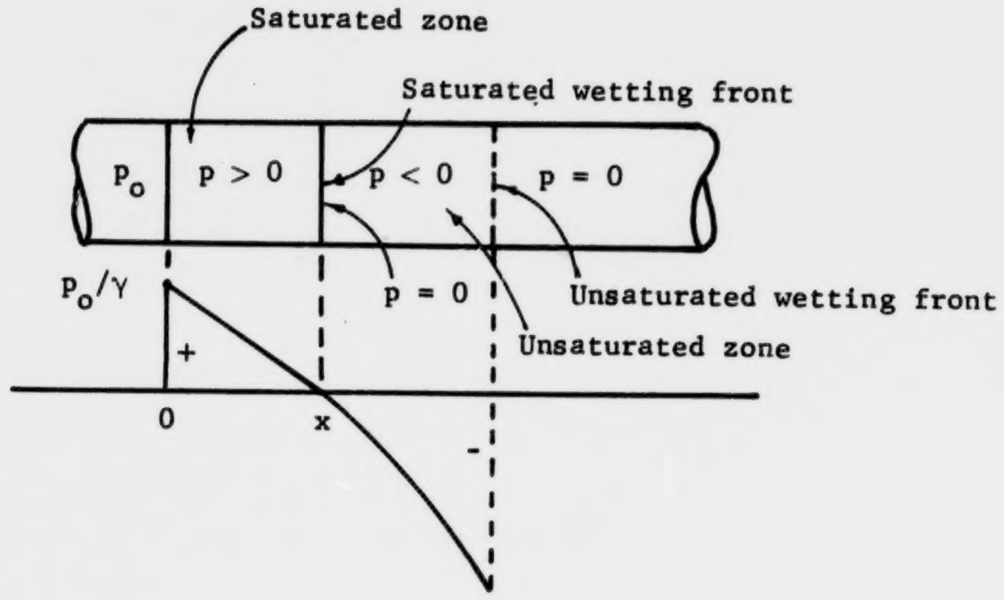


Figure 1.1 Horizontal Flow into a Column of Soil with Zero Critical-suction.

Figure 1.1 shows the case of a soil with zero critical suction.[1] In this case the pressure drops linearly from the applied pressure, p_o , to zero pressure at the saturated wetting front. Because of capillarity, some soils will exhibit a non-zero critical suction which results in negative pressures in the saturated region as shown in Fig. 1.2. The region of the saturated zone between the point of zero pressure and the interface between the saturated and unsaturated zones is referred to as the tension-saturated zone (Philip, 1958). Both the saturated and unsaturated wetting fronts are moving at different varying speeds. The flow in the saturated zone is governed by Darcy's law and the moisture movement in the unsaturated zone can be analyzed by applying the diffusion equation with certain modifications. It is assumed that the air flow resistance is negligible. Because of the moving saturated wetting front, it is necessary to utilize a transformation to change the moving boundary to a stationary one.

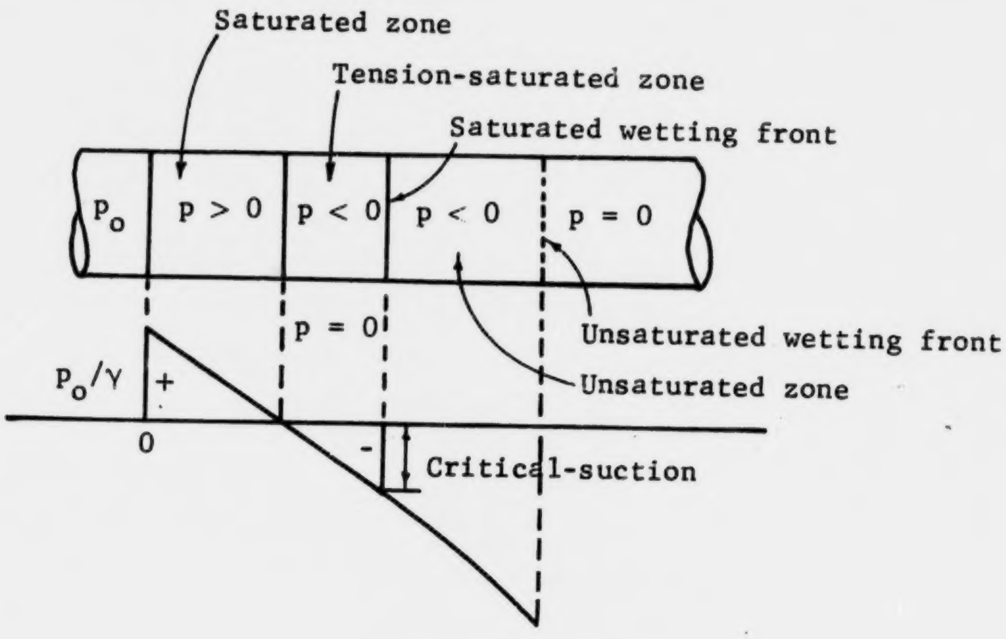

Saturated zone
Tension-saturated zone
Saturated wetting front
p_o
$p > 0$
$p < 0$
$p < 0$
$p = 0$
$p = 0$
Unsaturated wetting front
Unsaturated zone
p_o/γ
+
0
-
Critical-suction

Figure 1.2 Horizontal Flow into a Column of Soil with Non-zero Critical-suction.

---

[1] Critical suction represents the maximum negative pressure at which fully saturated conditions will exist in a soil. This is equivalent to the "air-entry value" commonly used by soil physicists, or to the "bubbling pressure" as used in the petroleum field.

The contents of the succeeding chapters are as follows. In Chapter 2, equations for saturated flow are developed for the cases of flow both with and without capillary forces acting at the wetting front. In Chapter 3, the equations for unsaturated flow and the parameters involved are presented and discussed. The Boltzmann transformation is used in Chapter 4 to transform the partial differential equation to an ordinary differential equation. Utilizing a transformation, in Chapter 5, the combined flow equations involving saturated flow followed by unsaturated flow are developed. In Chapter 6, the combined flow equations are solved by a numerical method using the Adams predictor-corrector formulas with the Runge-Kutta method for starting. Chapter 7 describes a series of experiments conducted under different applied pressures using silica flour as the medium. Gamma-ray attenuation equipment was used to provide a fast and accurate method of measuring the moisture profiles. The zero applied pressure experiment allows one to calculate the diffusivity. Using the calculated diffusivity with the experimentally-determined saturated hydraulic conductivity and a computed value of critical suction, the theoretical results are obtained corresponding to the different applied pressures used in the experiments. Thus, the validity of the approach can be rigorously tested. Discussion and conclusions are presented in Chapter 8.

CHAPTER 2

EQUATIONS FOR SATURATED FLOW

2.1 Darcy's Law

Darcy's law for flow in a porous medium may be written as

$$V = - Ki_o \tag{2.1}$$

where V = flow velocity, K = hydraulic conductivity, and i_o = hydraulic energy gradient. The V in Eq. (2.1) is defined as the flow rate per unit area. It is considerably smaller than the actual velocity of the fluid, which varies from point to point within the medium depending on the size, shape and orientation of the pores.

2.2 Range of Validity of Darcy's Law

It has long been empirically known that Darcy's law fails at sufficiently high flow rates. In Darcy's law inertia forces are neglected and the viscous forces dominate the flow. As velocity is increased, the inertia forces become significant and Darcy's law no longer applies. The transition from laminar to turbulent flow in a porous medium is characterized by a Reynolds number, N_R, which is a dimensionless ratio of inertia to viscous forces. Its general form is

$$N_R = \frac{Vd\rho}{\mu}$$

where V = flow velocity, d = average of diameters of particles, ρ = density of fluid, and μ = dynamic viscosity of fluid. The critical value of Reynolds number at which the flow in a porous medium changes from laminar to turbulent has been found by various investigators to range between 1 and 10. However, it is believed that the flow is laminar for $N_R < 1$, always turbulent for $N_R \gg 10$, and possibly laminar for $1 < N_R < 10$.

2.3 Horizontal Flow Without Capillary Forces

Consider horizontal flow into a homogeneous isotropic porous medium under the influence of an applied pressure. Assume the porous medium is completely saturated up to the wetting front and neglect capillarity. Figure 2.1 shows a simplified flow diagram and pressure distribution for such a situation.

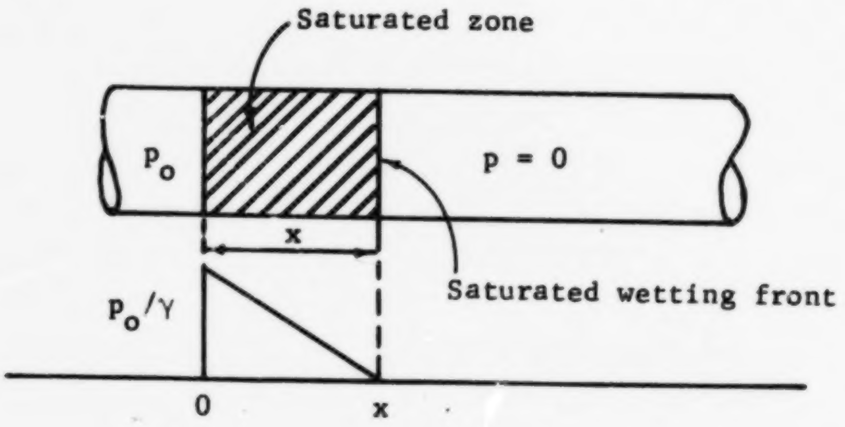


Figure 2.1 Horizontal Flow into a Soil Column Under an Applied Pressure with Zero Capillarity.

Let it be assumed that the applied pressure, p_o , is kept constant and applied uniformly to the surface of the porous medium. After some time t , the saturated wetting front will have moved a distance x . The pressure drops linearly from the applied pressure at the surface of the porous medium to zero at the saturated wetting front. The velocity of the saturated wetting front, $\frac{dx}{dt}$, is related to the flux through the surface by $V = n \frac{dx}{dt}$, where V is the flow velocity given by Darcy's law $V = -Ki_o$, and n is the porosity of the porous medium. Combining these two equations yields

$$n \frac{dx}{dt} = - Ki_o$$

where $$i_o = -\frac{p_o/\gamma}{x}$$

$$\therefore n\frac{dx}{dt} = K\frac{p_o/\gamma}{x}$$

or

$$x dx = \frac{K}{n}\frac{p_o}{\gamma} dt$$

The boundary conditions may be expressed as

$x = 0$ when $t = 0$

$x = x$ when $t = t$

Integrating between these limits yields

$$\frac{x^2}{2} = \frac{K}{n}\frac{p_o}{\gamma} t$$

or

$$x = \left(2\frac{K}{n}\frac{p_o}{\gamma} t\right)^{\frac{1}{2}} \tag{2.2}$$

which permits calculation of the position of the saturated wetting front for any given value of time. The flow velocity can be expressed as

$$V = -Ki_o = K\frac{p_o/\gamma}{x} \tag{2.3}$$

2.4 Horizontal Flow With Capillary Forces Concentrated at the Wetting Front

Let it be assumed that the capillary suction head, S, is acting uniformly at the saturated wetting front. Reference to Fig. 2.1 is very helpful in developing the flow equation. The head loss in the saturated zone is $p_o/\gamma + S$. Darcy's law combined with the equation of continuity becomes

$$\frac{dx}{dt} = \frac{K}{n} \frac{p_o/\gamma + S}{x}$$

which, after integration, yields

$$\frac{x^2}{2} = \frac{K}{n}\left(p_o/\gamma + S\right)t$$

or

$$x = \left[2 \frac{K}{n}\left(p_o/\gamma + S\right) t\right]^{\frac{1}{2}} \qquad (2.5)$$

The flow velocity can be expressed as

$$V = K \frac{p_o/\gamma + S}{x} \qquad (2.6)$$

CHAPTER 3

UNSATURATED FLOW IN SOILS

3.1 Equations Governing Unsaturated Flow

Flow through a porous medium must satisfy the equation of continuity provided that the medium is incompressible and homogeneous as shown in Fig. 3.1.

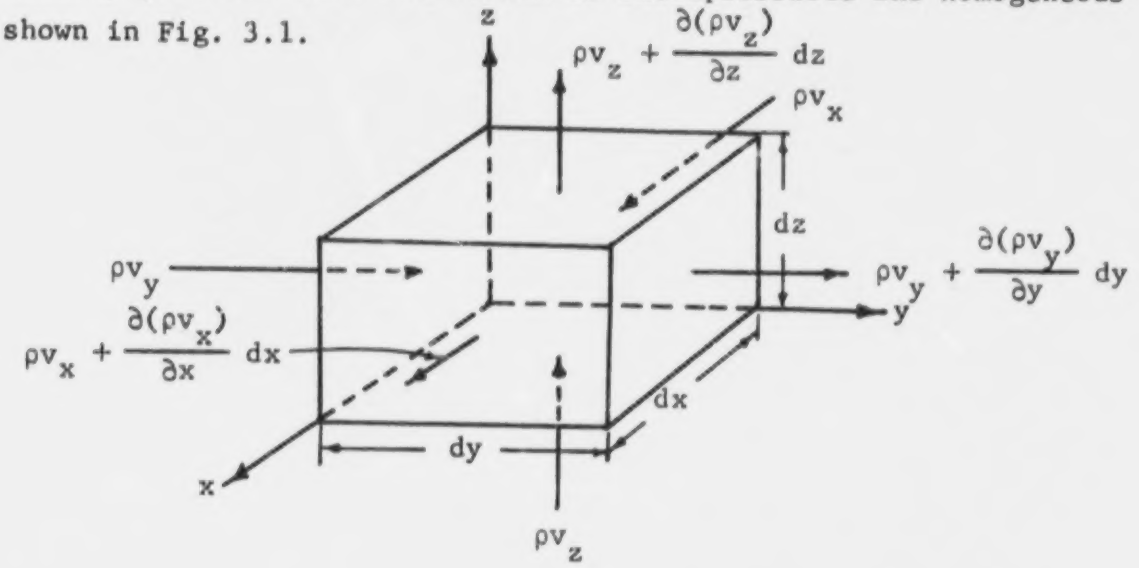

z
$\rho v_z + \frac{\partial(\rho v_z)}{\partial z} dz$
ρv_x
dz
ρv_y
$\rho v_y + \frac{\partial(\rho v_y)}{\partial y} dy$
y
$\rho v_x + \frac{\partial(\rho v_x)}{\partial x} dx$
dx
dy
x
ρv_z

Figure 3.1 Elementary Volume for Establishing Continuity Equations.

The equation of continuity may be expressed as

$$\frac{\partial \rho'}{\partial t} = - \left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} \right] \qquad (3.1)$$

where t is time, ρ' is the mass of fluid per total volume, ρ is fluid density, and v_x , v_y , v_z are the velocity components in x, y, and z directions. Expressed in vector form,

$$\frac{\partial \rho'}{\partial t} = - \nabla \cdot (\rho v)$$

Darcy's law may be expressed in vector form as

$$V = - K \nabla \varphi \qquad (3.2)$$

where φ is the total potential. Substituting Eq. (3.2) in Eq. (3.1) yields

$$\frac{\partial \rho'}{\partial t} = \nabla \cdot (\rho K \nabla \varphi) \qquad (3.3)$$

For the case of unsaturated flow, assuming Darcy's law is applicable, Eq. (3.3) can be written as

$$\frac{\partial(\rho\theta)}{\partial t} = \nabla \cdot (\rho K \nabla \varphi) \qquad (3.4)$$

where θ is the moisture content. If the fluid is of uniform composition and incompressible, ρ is constant, and Eq. (3.4) reduces to

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (K \nabla \varphi) \qquad (3.5)$$

which is the general partial differential equation for unsteady unsaturated flow of an incompressible fluid in a homogeneous and isotropic medium.

In unsaturated flow the total potential, φ, comprises the gravitational potential, z, which is positive upward, and the capillary potential, S, which is the capillary suction head with S expressed positively. Thus Eq. (3.5) may be written as

$$\frac{\partial \theta}{\partial t} = \nabla \cdot [K \nabla (S + z)] = \nabla \cdot [K \nabla S + K]$$

Hence,

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (K \nabla S) + \frac{\partial K}{\partial z} \qquad (3.6)$$

Equation (3.6) can be written in Cartesian coordinates as

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial S}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \frac{\partial S}{\partial y}\right) + \frac{\partial}{\partial z}\left(K \frac{\partial S}{\partial z}\right) + \frac{\partial K}{\partial z} \qquad (3.7)$$

When S and K are single-valued functions of θ, Eq. (3.7) becomes

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial S}{\partial\theta}\frac{\partial\theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \frac{\partial S}{\partial\theta}\frac{\partial\theta}{\partial y}\right) + \frac{\partial}{\partial z}\left(K \frac{\partial S}{\partial\theta}\frac{\partial\theta}{\partial z}\right) + \frac{\partial K(\theta)}{\partial z} \tag{3.8}$$

or

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial\theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[D(\theta)\frac{\partial\theta}{\partial y}\right] + \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial\theta}{\partial z}\right] + \frac{\partial K(\theta)}{\partial z} \tag{3.9}$$

where $D(\theta) = K \frac{\partial S}{\partial\theta}$ is defined as the diffusivity. Equation (3.9) sometimes is referred to as the diffusion equation.

For the special case of infiltration into a soil, Eq. (3.8) may be written as:

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial\theta}{\partial z}\right] - \frac{\partial K(\theta)}{\partial z} \tag{3.10}$$

In this case the sign of the second term is negative as z is defined as positive downward.

3.2 Basic Assumptions in Unsteady, Unsaturated Flow

Before attempting the solution and discussing the implications of the diffusion equation, it is important to make clear the basic assumptions upon which Eq. (3.9) was derived. The basic assumptions are

(a) Darcy's law is valid for unsaturated flow. Although the development of Darcy's law was the result of experimentation with saturated flow through filter sands, it has long been assumed to be applicable to unsaturated flow in the range of validity of Darcy's law. This was confirmed by the experiments of Childs and Collis-George (1950).

(b) In deriving the diffusion equation, we have implicitly made the assumption that the air in the medium offered no resistance to flow. The air originally in the medium is assumed to have a free passage to the atmosphere. Adrian (1964) has pointed out that air resistance may have a significant effect on infiltration when the escape of air is blocked. In such a case the air is trapped ahead of the wetting front resulting in an increase of air pressure in that region. In most practical situations free passage of air is possible so that concern about air resistance is usually unnecessary.

(c) The porous medium is homogeneous, i.e., any given property of the medium is independent of the x, y, and z coordinates. This is never true in practice as porosity and particle size of a medium vary from point to point. However, if one uses the mean value of porosity and particle size, satisfactory results can be obtained if the medium is relatively homogeneous.

(d) The porous medium is isotropic, i.e., any given property of the medium is independent of direction. This idealized situation is never encountered in practice. For the cases of horizontal absorption and infiltration where the flow is unidirectional, the lack of isotropy is irrelevant.

(e) The flow is isothermal. It has been found that variation in temperature will affect both the hydraulic conductivity and diffusivity. But for small variations in temperature the effect is negligible.

(f) The properties of the porous medium are not affected by biological or chemical phenomena.

3.3 Hydraulic Conductivity

In a soil not all of the liquid contributes to the flow. Below a certain moisture content θ_o there is no flow in an unsaturated soil. This θ_o is known as the threshold moisture content. It is the moisture content below which the liquid film continuum no longer exists. Only vapor movement occurs when moisture content is less than θ_o. Even in a saturated soil there may be some moisture held in place by molecular forces that does not contribute to the flow. This is known as the inactive moisture content θ_i. The moisture content at full saturation is defined as the saturated moisture content, θ_s, and is equal to the porosity of soil when expressed on a volumetric basis. Figure 3.2 illustrates, schematically, the relationship between threshold moisture content θ_o and inactive moisture content θ_i.

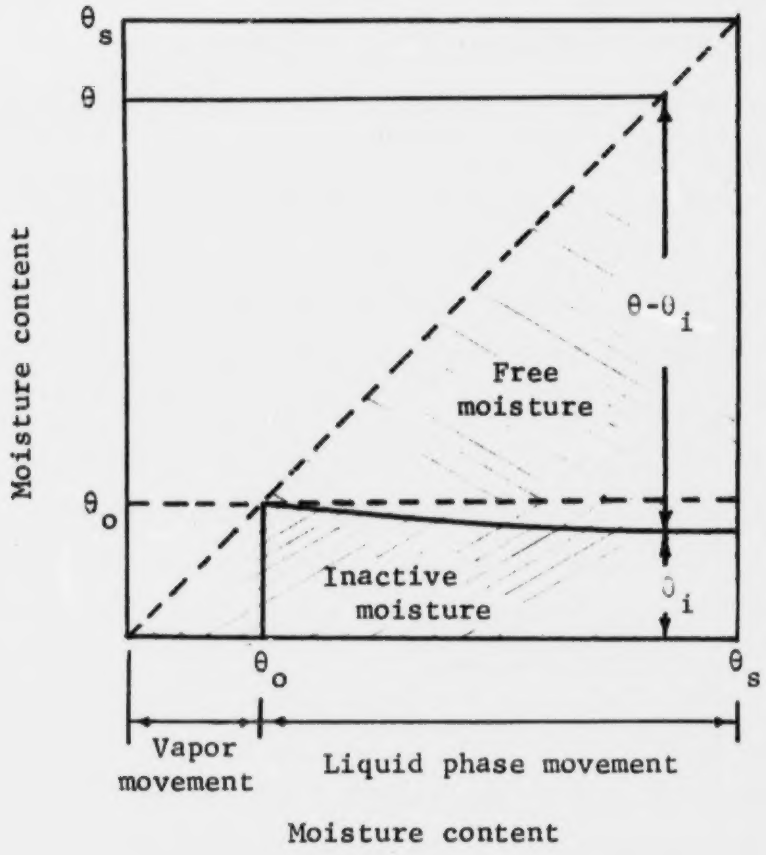

θ_s
θ
Moisture content
$\theta - \theta_i$
Free moisture
θ_o
Inactive moisture
θ_i
θ_o
θ_s
Vapor movement
Liquid phase movement
Moisture content

Figure 3.2 Schematic Representation of Relation Between Threshold Moisture Content θ_o and Inactive Moisture Content θ_i. (After Wang, 1963)

In most instances it is assumed that $\theta_o = \theta_i$; however, this is probably not true as shown in Fig. 3.2.

The dominant quantities affecting unsaturated flow are specific weight of water γ , dynamic viscosity μ , some characteristic length, say the average pore size of soil (which can be assumed proportional to a representative grain diameter d), the porosity n , and the moisture content in the range of liquid movement $(\theta - \theta_o)$. Hence the hydraulic conductivity K may be expressed as:

$$K = C(\gamma)^{x_1} (\mu)^{x_2} (d)^{x_3} (n)^{x_4} (\theta - \theta_o)^{x_5} \tag{3.11}$$

Expressing all quantities in one fundamental system: mass, length, time, i.e., [M] , [L] , [T] gives

$$[M]^o[L][T]^{-1} = C[M/L^2T^2]^{x_1} [M/LT]^{x_2} [L]^{x_3} \tag{3.12}$$

It should be noted that n and $(\theta - \theta_o)$ are dimensionless. The above functional relationship is dimensionally correct if the exponents of [M] , [L] , [T] on both sides are the same. Hence three equations in x_1 , x_2 , and x_3 are obtained:

$$[M] \rightarrow 0 = x_1 + x_2$$

$$[L] \rightarrow 1 = -2x_1 - x_2 + x_3$$

$$[T] \rightarrow -1 = -2x_1 - x_2$$

This system of equations has a unique solution, namely, $x_1 = 1$, $x_2 = -1$, $x_3 = 2$. Therefore

$$K = C \frac{\gamma}{\mu} d^2 (n)^q (\theta - \theta_o)^r \tag{3.13}$$

where C , q , and r are three constants to be determined experimentally. For saturated moisture content, $K = K_s$ and

$$K_s = C \frac{\gamma}{\mu} d^2 (n)^q (\theta_s - \theta_o)^r \tag{3.14}$$

Dividing Eq. (3.13) by (3.14) yields

$$\frac{K}{K_s} = \frac{(\theta - \theta_o)^r}{(\theta_s - \theta_o)^r}$$

or

$$K = K_s\left(\frac{\theta - \theta_o}{\theta_s - \theta_o}\right)^r \tag{3.15}$$

For most soils, the range of r is between 3 and 6. Figure (3.3) shows a typical relation between K and θ .

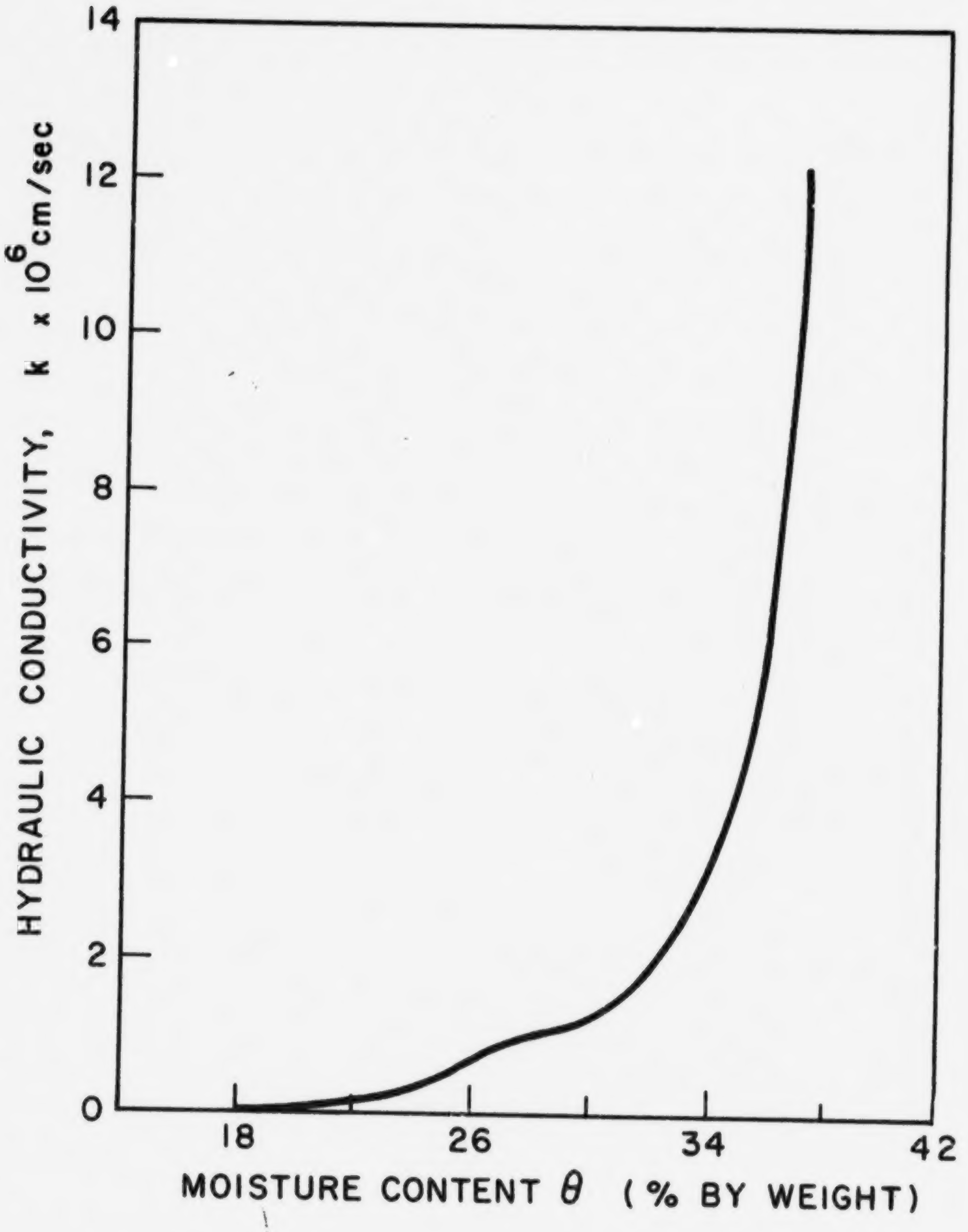


Figure 3.3 Relation between Hydraulic Conductivity and Moisture Content for Yolo Light Clay (After Moore, 1939).

3.4 Diffusivity

As defined in Section 3.1, diffusivity is

$$D(\theta) = K(\theta) \frac{\partial S(\theta)}{\partial \theta} \quad .$$

Numerous empirical expressions have been derived by various investigators (Ohmstede, 1957) to relate the diffusivity to moisture content. For the present work the exponential relationship will be used. Its general form is

$$D(\theta) = \alpha e^{\beta\theta} \tag{3.18}$$

where α and β are two constants to be determined experimentally. As predicted by Eq. (3.18) the plot of diffusivity versus moisture content on a semilogarithmic paper yields a straight line as shown in Fig. 3.4.

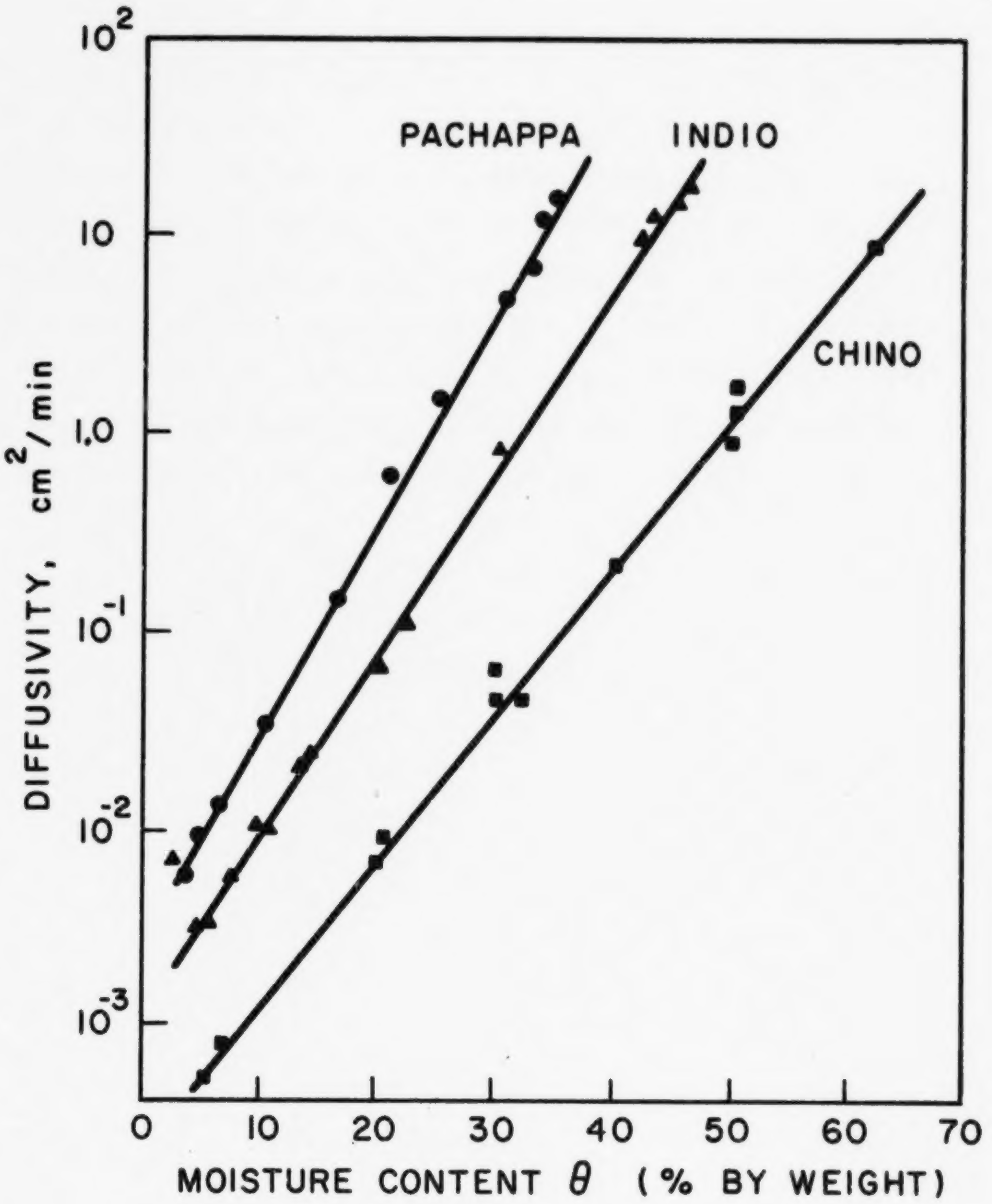


Figure 3.4 Diffusivity as a Function of Moisture Content for Three Soils (After Ohmstede, 1957).

3.5 Hysteresis Phenomena

During infiltration the moisture content of a medium increases with time. That is, it is a wetting situation. For such a case the soil moisture tension (or capillary force) is a unique function of moisture content. However, situations involving decreasing moisture contents, e.g., the drying condition, occur quite commonly and are of equal importance to engineers. Unfortunately, the relationship between soil tension and moisture content is not the same on wetting condition as on drying. It has been shown that the soil tension is much lower during the wetting condition. This phenomenon is known as hysteresis. Because of this, one must be careful when dealing with problems involving both wetting and drying. Figure 3.5 shows the relation between soil moisture tension and moisture content.

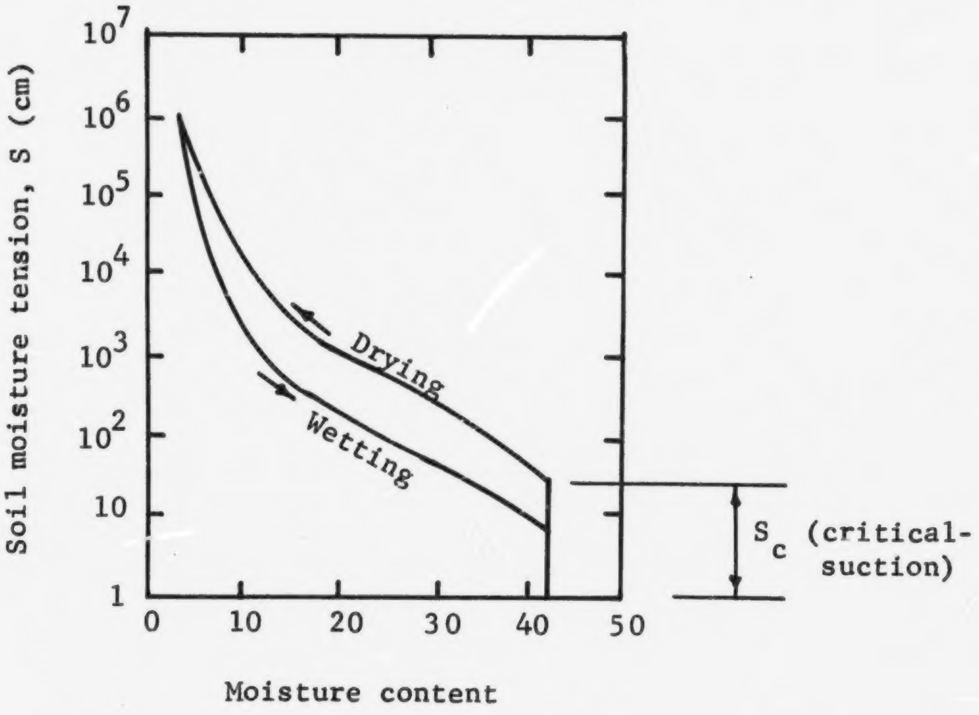


Figure 3.5 Moisture-tension Curves for a Typical Soil.

3.6 Critical-suction Value

Critical-suction value (or air-entry value) corresponds to the suction at which air enters an initially saturated porous medium. It may be viewed as being approximately equal to the maximum suction at which a medium will maintain complete saturation. In most research to date dealing with soil-moisture movement, the non-zero critical-suction property of soils has been neglected. The "diffusion of ψ" approach by Philip (1958b) is one of the few places where the non-zero critical-suction value is discussed. Figure 3.5 illustrates the relation between moisture content and suction for a soil that has a non-zero critical-suction value. It should be noted that not every soil exhibits a distinct non-zero critical-suction value. If a soil is disturbed and repacked, the critical-suction value will change. The critical-suction value is larger on drying condition than on wetting. There are several experimental methods for determining critical-suction value, none of which are entirely satisfactory.

As shown in Fig. 1.2, the tension-saturated zone is caused by the non-zero critical-suction value. A further discussion of this phenomenon is presented in Secs. 5.3, 7.8 and 7.10.

CHAPTER 4

TRANSFORMATION OF THE EQUATION FOR HORIZONTAL UNSATURATED FLOW TO AN ORDINARY DIFFERENTIAL EQUATION

The equation for infiltration, Eq. (3.10), studied by many investigators, is

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(D\,\frac{\partial\theta}{\partial z}\right) - \frac{\partial K}{\partial z} \tag{4.1}$$

with boundary conditions

$$\left.\begin{array}{lll} \theta = \theta_n \quad , & t = 0 \quad , & z > 0 \\ \theta = \theta_s \quad , & z = 0 \quad , & t \geq 0 \end{array}\right\}$$

where θ_n = the initial moisture content, θ_s = the saturated moisture content, and z is taken as positive downward.

Let us now consider horizontal flow and introduce the Boltzmann transformation. Neglecting gravity and placing the z-axis in the horizontal plane, Eq. (4.1) becomes

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(D\,\frac{\partial\theta}{\partial z}\right) \tag{4.2}$$

The Boltzmann transformation may be employed as follows:

$$\varphi = z/(t)^{\frac{1}{2}}$$

$$\frac{\partial\varphi}{\partial t} = -\,\frac{1}{2}\,zt^{-\,3/2} = -\,\frac{\varphi}{2t}$$

$$\frac{\partial\varphi}{\partial z} = 1/(t)^{\frac{1}{2}} \quad , \quad \frac{\partial^2\varphi}{\partial z^2} = 0$$

$$\frac{\partial\theta}{\partial z} = \frac{\partial\theta}{\partial\varphi}\cdot\frac{\partial\varphi}{\partial z} = \frac{\partial\theta}{\partial\varphi}\,\frac{1}{(t)^{\frac{1}{2}}}$$

Substituting the above derivatives into Eq. (4.2)

$$\frac{\partial\theta}{\partial\varphi} \cdot \frac{\partial\varphi}{\partial t} = \frac{\partial}{\partial\varphi}\left(D \frac{\partial\theta}{\partial\varphi} \frac{\partial\varphi}{\partial t} \right) \frac{\partial\varphi}{\partial z}$$

$$\frac{\partial\theta}{\partial\varphi} \frac{\partial\varphi}{\partial t} = \frac{\partial D}{\partial\varphi} \frac{\partial\theta}{\partial\varphi} \left(\frac{\partial\varphi}{\partial z} \right)^2 + D \frac{\partial^2\theta}{\partial\varphi^2} \left(\frac{\partial\varphi}{\partial z} \right)^2 + D \frac{\partial\theta}{\partial\varphi} \frac{\partial}{\partial\varphi} \left(\frac{\partial\varphi}{\partial z} \right) \frac{\partial\varphi}{\partial z}$$

$$\frac{\partial\theta}{\partial\varphi} \frac{\partial\varphi}{\partial t} = \left(\frac{\partial D}{\partial\varphi} \frac{\partial\theta}{\partial\varphi} + D \frac{\partial^2\theta}{\partial\varphi^2} \right) \left(\frac{\partial\varphi}{\partial z} \right)^2$$

$$\frac{\partial\theta}{\partial\varphi} \frac{\partial\varphi}{\partial t} = \frac{\partial}{\partial\varphi} \left(D \frac{\partial\theta}{\partial\varphi} \right) \left(\frac{\partial\varphi}{\partial z} \right)^2$$

or

$$\frac{\partial\theta}{\partial\varphi} \left(- \frac{\varphi}{2t} \right) = \frac{\partial}{\partial\varphi} \left(D \frac{\partial\theta}{\partial\varphi} \right) \left(\frac{\partial\varphi}{\partial z} \right)^2$$

or

$$\left(- \frac{\varphi}{2t} \right) \frac{\partial\theta}{\partial\varphi} = \frac{\partial}{\partial\varphi} \left(D \frac{\partial\theta}{\partial\varphi} \right) \left(\frac{1}{(t)^{\frac{1}{2}}} \right)^2$$

Finally,

$$- \frac{\varphi}{2} \frac{d\theta}{d\varphi} = \frac{d}{d\varphi} \left(D \frac{d\theta}{d\varphi} \right)$$

or

$$\frac{d}{d\varphi} \left(D \frac{d\theta}{d\varphi} \right) + \frac{\varphi}{2} \frac{d\theta}{d\varphi} = 0 \qquad (4.3)$$

The above equation is subject to the following:

Initial conditions: $\theta = \theta_n$, $t = 0$, $\varphi = \frac{z}{(t)^{\frac{1}{2}}} = \infty$

Boundary conditions: $\theta = \theta_s$, $z = 0$, $\varphi = \frac{z}{(t)^{\frac{1}{2}}} = 0$

Substituting $D = \alpha e^{\beta\theta}$ into Eq. (4.3) gives

$$\frac{d}{d\varphi}\left(\alpha e^{\beta\theta}\frac{d\theta}{d\varphi}\right) + \frac{\varphi}{2}\frac{d\theta}{d\varphi} = 0$$

$$\frac{d}{d\varphi}\left(\alpha e^{\beta\theta}\right)\theta' + \alpha e^{\beta\theta}\,\theta'' + \frac{\varphi}{2}\theta' = 0$$

where

$$\theta' = \frac{d\theta}{d\varphi} \qquad \text{and} \qquad \theta'' = \frac{d^2\theta}{d\varphi^2}$$

It can be further reduced to

$$\alpha\beta e^{\beta\theta}(\theta')^2 + \alpha e^{\beta\theta}\theta'' + \frac{\varphi}{2}\theta' = 0$$

or

$$\alpha e^{\beta\theta}\theta'' + \alpha\beta e^{\beta\theta}(\theta')^2 + \frac{\varphi}{2}\theta' = 0 \tag{4.4}$$

Making a change in variables

$$\begin{aligned} x &= \frac{\varphi}{(\alpha)^{\frac{1}{2}}} \\ y &= e^{\beta\theta} \quad ; \quad \theta = \frac{1}{\beta}\ln y \end{aligned} \tag{4.5}$$

then Eq. (4.4) becomes

$$\alpha y\theta'' + \alpha\beta y(\theta')^2 + \frac{\varphi}{2}\theta' = 0 \tag{4.6}$$

$$\theta' = \frac{d\theta}{d\varphi} = \frac{d}{dx}\left(\frac{1}{\beta}\ln y\right)\frac{dx}{d\varphi} = \frac{1}{\beta}\frac{y'}{y}\frac{1}{(\alpha)^{\frac{1}{2}}}$$

$$\theta'' = \frac{d^2\theta}{d\varphi^2} = \frac{1}{\beta(\alpha)^{\frac{1}{2}}} \frac{d(y'/y)}{dx} \frac{dx}{d\varphi} = \frac{1}{\alpha\beta}\left[\frac{y'}{y} - \frac{(y')^2}{y^2}\right]$$

Substituting the above derivations into Eq. (4.6) gives

$$\alpha y \frac{1}{\alpha\beta}\left[\frac{y''}{y} - \frac{(y')^2}{y^2}\right] + \alpha\beta y \frac{1}{\alpha\beta^2} \frac{(y')^2}{y^2} + \frac{(\alpha)^{\frac{1}{2}}}{2} \cdot x \cdot \frac{y'}{(\alpha)^{\frac{1}{2}}\beta y} = 0$$

$$\frac{y}{\beta}\left[\frac{y''}{y} - \frac{(y')^2}{y^2}\right] + \frac{y}{\beta} \frac{(y')^2}{y^2} + \frac{x}{2\beta} \frac{y'}{y} = 0$$

$$y'' - \frac{(y')^2}{y} + \frac{(y')^2}{y} + \frac{xy'}{2y} = 0$$

$$y'' + \frac{x}{2} \frac{y'}{y} = 0$$

or

$$y'' = -\, y' \frac{x}{2y} \tag{4.7}$$

subject to the transformed boundary conditions

$$\left.\begin{array}{llll} \theta = \theta_n & x = \dfrac{\varphi}{(\alpha)^{\frac{1}{2}}} = \dfrac{z}{(\alpha t)^{\frac{1}{2}}} = \infty & , & y = y_n = e^{\beta\theta_n} \\[2ex] \theta = \theta_s & x = \dfrac{z}{(\alpha t)^{\frac{1}{2}}} = 0 & , & y = y_s = e^{\beta\theta_s} \end{array}\right\} \quad (4.8)$$

At a given time t_1, the wetting front would have moved a finite distance z_1. Beyond z_1 at this time, the moisture content is constant and equal to θ_n out to infinity. For the finite values of t_1 and z_1, x_1 is also finite by the relation

$$x_1 = z_1/(\alpha t_1)^{\frac{1}{2}}$$

With this argument, we can conclude that

$$\left.\begin{aligned} x &= x_1 = \frac{z_1}{(\alpha t_1)^{\frac{1}{2}}} \quad , \quad y = e^{\beta\theta_n} \\ x &= 0 \quad , \quad y = e^{\beta\theta_s} \end{aligned}\right\} \qquad (4.9)$$

The flow velocity, V, can be written as

$$V = \frac{d}{dt}\int_{\theta_n}^{\theta_s} z \, d\theta \qquad (4.10)$$

or

$$V = \frac{d}{dt}\int_{\theta_n}^{\theta_s} \varphi(t)^{\frac{1}{2}} \, d\theta \qquad (4.11)$$

and

$$V = \frac{1}{2(t)^{\frac{1}{2}}}\int_{\theta_n}^{\theta_s} \varphi \, d\theta \qquad (4.12)$$

From Eq. (4.5), we have

$$\varphi = (\alpha)^{\frac{1}{2}}x \quad ,$$

$$\theta = \frac{1}{\beta}\,\ell n\ y \quad ; \quad d\theta = \frac{1}{\beta}\frac{dy}{y}$$

Substituting these relations into Eq. (4.12) yields

$$V = \frac{1}{2(t)^{\frac{1}{2}}}\int_{y_n}^{y_s} (\alpha)^{\frac{1}{2}}x\left(\frac{1}{\beta}\frac{dy}{y}\right) = \frac{(\alpha)^{\frac{1}{2}}}{2\beta(t)^{\frac{1}{2}}}\int_{y_n}^{y_s}\frac{x}{y}\,dy \qquad (4.13)$$

From Eq. (4.7), we have

$$\frac{x}{y} = -\,2\,\frac{y''}{y'}$$

Substituting the above relation into Eq. (4.13) and carrying out the integration, we obtain

$$V = -\,\frac{(\alpha)^{\frac{1}{2}}}{\beta(t)^{\frac{1}{2}}}\,y'_{z_s} \qquad (4.14)$$

It should be noted that $y'_{z_n} = 0$. The accumulative absorption is $\int_0^t V\,dt$.

CHAPTER 5

COMBINED FLOW EQUATION FOR UNSATURATED FLOW FOLLOWED BY SATURATED FLOW

5.1 Horizontal One-Dimensional Flow Into a Column of Soil

Figure 5.1 illustrates the flow into a horizontal semi- infinite column of soil, initially at moisture content θ_n , with flow introduced under a certain applied hydrostatic pressure p_o .

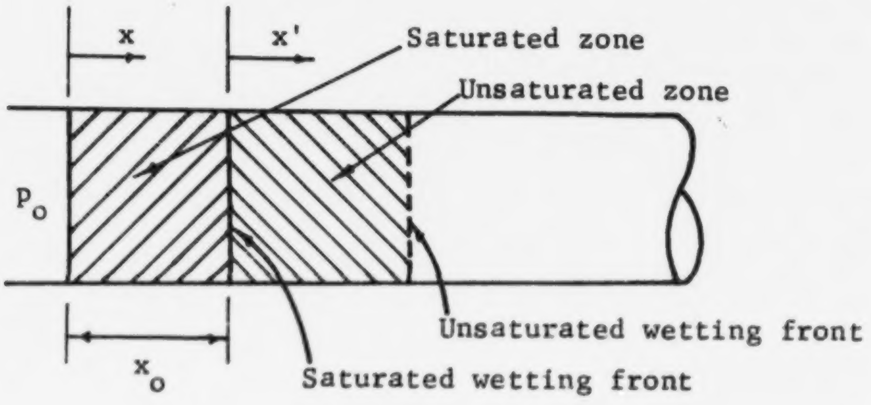

x
x'
Saturated zone
Unsaturated zone
p_o
Unsaturated wetting front
x_o
Saturated wetting front

Figure 5.1 Schematic Representation of Horizontal Flow into a Column of Soil.

It should be noted that both the saturated and unsaturated wetting fronts are moving, but at different rates. As a consequence, in order to apply the expressions for unsaturated flow developed in Chapter 4, it is necessary to transform the moving boundary defined by the saturated wetting front into a stationary boundary. It has been demonstrated in Eq. (2.2) that the penetration of the saturated wetting front is proportional to the square root of time. For unsaturated flow, the Boltzmann transformation implies that a plane of constant moisture content also advances a distance proportional to the square root of time. In the case of unsaturated flow followed by a saturated flow, the distance of penetration of the saturated wetting front can also be assumed to be proportional to the square root of time. Thus, we can write

$$x_o = c(t)^{\frac{1}{2}} \tag{5.1}$$

where x_o is the distance of penetration of the saturated wetting front and c is a constant of proportionality.

Let x' be a moving coordinate system with its origin at the saturated wetting front, i.e., at $x = x_o$ as shown in Fig. 5.1. Using the transformation

$$\left.\begin{array}{l} x' = x - x_o \\ t' = t \end{array}\right\} \tag{5.2}$$

where x_o defines the position of the saturated wetting front, we obtain

$$\left.\begin{array}{l} x' = x - c(t)^{\frac{1}{2}} \\ t' = t \end{array}\right\} \tag{5.3}$$

Rewriting the diffusion equation,

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\,\frac{\partial\theta}{\partial x}\right] \tag{4.2}$$

subject to the following conditions:

$$\left.\begin{array}{lll} \theta = \theta_n , & t = 0 , & x > 0 \\ \theta = \theta_s , & x = 0 , & t \geq 0 \end{array}\right\}$$

and using Eq. (5.3), the following derivatives are obtained,

$$\frac{\partial x'}{\partial t} = - \frac{c}{2(t)^{\frac{1}{2}}} \quad ,$$

$$\frac{\partial x'}{\partial x} = 1 \quad ,$$

$$\frac{\partial t'}{\partial t} = 1 \quad ,$$

$$\frac{\partial t'}{\partial x} = 0 \quad ,$$

$$\frac{\partial \theta(x,t)}{\partial x} = \frac{\partial \theta(x',t')}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial \theta(x',t')}{\partial t'} \frac{\partial t'}{\partial x} = \frac{\partial \theta(x',t')}{\partial x'} \quad ,$$

$$\frac{\partial \theta(x,t)}{\partial t} = \frac{\partial \theta(x',t')}{\partial x'} \frac{\partial x'}{\partial t} + \frac{\partial \theta(x',t')}{\partial t'} \frac{\partial t'}{\partial t}$$

$$= - \frac{c}{2(t)^{\frac{1}{2}}} \frac{\partial \theta(x',t')}{\partial x'} + \frac{\partial \theta(x',t')}{\partial t'}$$

Substituting the above derivatives into Eq. (4.2) and writing t for t' yields

$$- \frac{c}{2(t)^{\frac{1}{2}}} \frac{\partial \theta}{\partial x'} + \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x'} \left(D \frac{\partial \theta}{\partial x'} \right) \tag{5.4}$$

subject to:

$$\left. \begin{aligned} \theta &= \theta_n \quad , \quad t = 0 \quad , \quad x' > 0 \\ \theta &= \theta_s \quad , \quad x' = 0 \quad , \quad t \geq 0 \end{aligned} \right\} \tag{5.5}$$

Using the Botlzmann transformation, $\varphi = x'/(t)^{\frac{1}{2}}$, the following derivatives are obtained,

$$\frac{\partial \varphi}{\partial t} = \frac{1}{2} x' t^{-3/2} = - \frac{\varphi}{2t} \quad ,$$

$$\frac{\partial\varphi}{\partial x'} = \frac{1}{(t)^{\frac{1}{2}}} \quad , \quad \frac{\partial^2\varphi}{\partial x'^2} = 0 \quad ,$$

$$\frac{\partial\theta}{\partial x'} = \frac{\partial\theta}{\partial\varphi}\frac{\partial\varphi}{\partial x'} = \frac{\partial\theta}{\partial\varphi}\frac{1}{(t)^{\frac{1}{2}}} \quad .$$

Substituting the above derivatives into Eq. (5.4) yields

$$-\frac{c}{2(t)^{\frac{1}{2}}}\frac{\partial\theta}{\partial\varphi}\frac{\partial\varphi}{\partial x'} + \frac{\partial\theta}{\partial\varphi}\frac{\partial\varphi}{\partial t} = \frac{\partial}{\partial\varphi}\left(D\frac{\partial\theta}{\partial\varphi}\frac{\partial\varphi}{\partial x'}\right)\frac{\partial\varphi}{\partial x'}$$

$$-\frac{c}{2(t)^{\frac{1}{2}}}\frac{\partial\theta}{\partial\varphi}\frac{\partial\varphi}{\partial x'} + \frac{\partial\theta}{\partial\varphi}\frac{\partial\varphi}{\partial t} = \frac{\partial D}{\partial\varphi}\frac{\partial\theta}{\partial\varphi}\left(\frac{\partial\varphi}{\partial x'}\right)^2 + D\frac{\partial^2\theta}{\partial\varphi^2}\left(\frac{\partial\varphi}{\partial x'}\right)^2$$

$$+ D\frac{\partial\theta}{\partial\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\partial\varphi}{\partial x'}\right)\frac{\partial\varphi}{\partial x'}$$

$$-\frac{c}{2(t)^{\frac{1}{2}}}\frac{\partial\theta}{\partial\varphi}\frac{\partial\varphi}{\partial x'} + \frac{\partial\theta}{\partial\varphi}\frac{\partial\varphi}{\partial t} = \frac{\partial}{\partial\varphi}\left(D\frac{\partial\theta}{\partial\varphi}\right)\left(\frac{\partial\varphi}{\partial x'}\right)^2$$

$$-\frac{c}{2(t)^{\frac{1}{2}}}\frac{\partial\theta}{\partial\varphi}\left(\frac{1}{(t)^{\frac{1}{2}}}\right) + \frac{\partial\theta}{\partial\varphi}\left(-\frac{\varphi}{2t}\right) = \frac{\partial}{\partial\varphi}\left(D\frac{\partial\theta}{\partial\varphi}\right)\left(\frac{1}{(t)^{\frac{1}{2}}}\right)^2$$

$$-\frac{c}{2}\frac{d\theta}{d\varphi} - \frac{\varphi}{2}\frac{d\theta}{d\varphi} = \frac{d}{d\varphi}\left(D\frac{d\theta}{d\varphi}\right)$$

Finally,

$$-\left(\frac{c+\varphi}{2}\right)\frac{d\theta}{d\varphi} = \frac{d}{d\varphi}\left(D\frac{\partial\theta}{\partial\varphi}\right) \qquad (5.6)$$

subject to the transformed conditions:

$$\left.\begin{array}{ll} \theta = \theta_n & , \quad \varphi = x'/(t)^{\frac{1}{2}} = \infty \\ \theta = \theta_s & , \quad \varphi = x'/(t)^{\frac{1}{2}} = 0 \end{array}\right\} \qquad (5.7)$$

Substituting $D = \alpha e^{\beta\theta}$ into Eq. (5.6) gives

$$-\left(\frac{c + \varphi}{2}\right)\frac{d\theta}{d\varphi} = \frac{d}{d\varphi}\left[\alpha e^{\beta\theta}\,\frac{\partial\theta}{\partial\varphi}\right] \tag{5.8}$$

or

$$\alpha e^{\beta\theta}\,\frac{d^2\theta}{d\varphi^2} + \alpha\beta e^{\beta\theta}\left(\frac{d\theta}{d\varphi}\right)^2 + \left(\frac{c + \varphi}{2}\right)\frac{d\theta}{d\varphi} = 0 \tag{5.9}$$

Making a change in variables,

$$\begin{aligned} u &= \frac{\varphi}{(\alpha)^{\frac{1}{2}}} + \frac{c}{(\alpha)^{\frac{1}{2}}} \\ v &= e^{\beta\theta} \quad ; \quad \theta = \frac{1}{\beta}\,\ell n\ v \end{aligned} \tag{5.10}$$

then Eq. (5.9) becomes

$$\alpha v\,\frac{d^2\theta}{d\varphi^2} + \alpha\beta v\left(\frac{d\theta}{d\varphi}\right)^2 + \left(\frac{c + \varphi}{2}\right)\frac{d\theta}{d\varphi} = 0 \tag{5.11}$$

and

$$\frac{d\theta}{d\varphi} = \frac{d}{du}\left(\frac{1}{\beta}\,\ell n\ v\right)\frac{du}{d\varphi} = \frac{1}{\beta}\,\frac{v'}{v}\,\frac{1}{(\alpha)^{\frac{1}{2}}}$$

$$\frac{d^2\theta}{d\varphi^2} = \frac{2}{\beta(\alpha)^{\frac{1}{2}}}\,\frac{d(v'/v)}{du}\,\frac{du}{d\varphi} = \frac{1}{\alpha\beta}\left[\frac{v''}{v} - \frac{(v')^2}{v^2}\right] ,$$

where $v' = \frac{dv}{du}$ and $v'' = \frac{d^2v}{du^2}$

Substituting these derivatives back into Eq. (5.11), we have

$$\alpha v\,\frac{1}{\alpha\beta}\left[\frac{v''}{v} - \frac{(v')^2}{v^2}\right] + \alpha\beta v\,\frac{1}{\alpha\beta^2}\,\frac{(v')^2}{v^2} + \left(\frac{c + x(\alpha)^{\frac{1}{2}} - c}{2}\right)\frac{v'}{(\alpha)^{\frac{1}{2}}\beta v} = 0$$

Finally,

$$v'' = -\,v'\left(\frac{u}{2v}\right) \tag{5.12}$$

subject to:

$$\left.\begin{array}{lll} \theta = \theta_n , & u = \infty , & v = v_n = e^{\beta\theta_n} \\ \theta = \theta_s , & u = \frac{c}{(\alpha)^{\frac{1}{2}}} , & v = v_s = e^{\beta\theta_s} \end{array}\right\} \tag{5.13}$$

The flow velocity, V, can be expressed as

$$V = \frac{d}{dt}\int_{\theta_n}^{\theta_s} x \, d\theta \tag{5.14}$$

or

$$V = \frac{d}{dt}\left[(\theta_s - \theta_n)x_o + \int_{\theta_n}^{\theta_s} x' \, d\theta\right] \tag{5.15}$$

Substituting $x_o = c(t)^{\frac{1}{2}}$ and $x' = \varphi(t)^{\frac{1}{2}}$ into Eq. (5.15), V can be reduced to

$$V = \frac{c}{2(t)^{\frac{1}{2}}}(\theta_s - \theta_n) + \frac{1}{2(t)^{\frac{1}{2}}}\int_{\theta_n}^{\theta_s} \varphi \, d\theta \tag{5.16}$$

From Eq. (5.10), we have

$$\varphi = u(\alpha)^{\frac{1}{2}} - c \quad ,$$

$$\theta = \frac{1}{\beta} \ln v \quad ; \quad d\theta = \frac{1}{\beta}\frac{dv}{v}$$

Substituting these relations into Eq. (5.16) yields

$$V = \frac{c}{2(t)^{\frac{1}{2}}}(\theta_s - \theta_n) + \frac{1}{2(t)^{\frac{1}{2}}}\int_{v_n}^{v_s} \left(u(\alpha)^{\frac{1}{2}} - c\right)\frac{1}{\beta}\frac{dv}{v}$$

or

$$V = \frac{c}{2(t)^{\frac{1}{2}}} (\theta_s - \theta_n) + \frac{(\alpha)^{\frac{1}{2}}}{2\beta(t)^{\frac{1}{2}}} \int_{v_n}^{v_s} \frac{u}{v}\, dv - \frac{c}{2\beta(t)^{\frac{1}{2}}} \int_{v_n}^{v_s} \frac{dv}{v}$$

Substituting $\frac{u}{v} = -2\frac{v''}{v'}$, which is obtained from Eq. (5.12), V can be written as

$$V = \frac{c}{2(t)^{\frac{1}{2}}} (\theta_s - \theta_n) + \frac{(\alpha)^{\frac{1}{2}}}{2\beta(t)^{\frac{1}{2}}} \int_{v_n}^{v_s} \left(-2\frac{v''}{v'}\right) dv$$

$$- \frac{c}{2\beta(t)^{\frac{1}{2}}} \int_{v_n}^{v_s} \frac{dv}{v} \tag{5.17}$$

By carrying out the integrations, we obtain

$$V = \frac{c}{2(t)^{\frac{1}{2}}} (\theta_s - \theta_n) - \frac{(\alpha)^{\frac{1}{2}}}{\beta(t)^{\frac{1}{2}}} \left[v'(u_s) - v'(u_n) \right]$$

$$- \frac{c}{2\beta(t)^{\frac{1}{2}}} \left[\ell n\, v_s - \ell n\, v_n \right]$$

It can be further reduced to

$$V = -\frac{(\alpha)^{\frac{1}{2}}}{\beta(t)^{\frac{1}{2}}} v'(u_s) \tag{5.18}$$

It should be noted that $v'(u_n) = 0$.

The flow velocity can also be obtained by applying Darcy's law to the saturated zone.

$$V = K_s \frac{P_o/\gamma}{x_o} = K_s \frac{P_o/\gamma}{c(t)^{\frac{1}{2}}} \tag{5.19}$$

Equating Eqs. (5.18) and (5.19) yields

$$c = \frac{-(p_o/\gamma)\ K_s\beta}{(\alpha)^{\frac{1}{2}}v'(u_s)} \tag{5.20}$$

Now the differential equation can be written as

$$v'' = -\ v'\left(\frac{u}{2v}\right) \tag{5.21}$$

subject to:

$$\left.\begin{array}{ll} u = \infty , & v = v_n = e^{\beta\theta_n} \\ u = \dfrac{-(p_o/\gamma)\ K_s\beta}{\alpha v'(u_s)} , & v = v_s = e^{\beta\theta_s} \end{array}\right\} \tag{5.22}$$

At a given time t_1 the wetting front would have moved to a finite distance x_1' . Beyond x_1' at this time, the moisture is constant (θ_n) out to infinity. For the finite values of t_1 and x_1' , u_1 is also finite by the following relation

$$u_1 = \frac{x_1'}{(\alpha t_1)^{\frac{1}{2}}} + \frac{c}{(\alpha)^{\frac{1}{2}}}$$

Thus, Eq. (5.22) can be written as

$$\left.\begin{array}{ll} u = u_1 , & v = v_n = e^{\beta\theta_n} \\ u = u_s = \dfrac{-(p_o/\gamma)\ K_s\beta}{\alpha v'(u_s)} , & v = v_s = e^{\beta\theta_s} \end{array}\right\} \tag{5.23}$$

5.2 Alternate Development of Flow Equations

Another method parallel to Philip's diffusion approach, analogous to heat flow, can also be used to develop the combined flow equation. Figure 5.2 shows the flow into a horizontal, semi-infinite column of uniform soil initially at a uniform moisture content θ_n and which from time zero on has a certain applied hydrostatic pressure p_o maintained at one end.

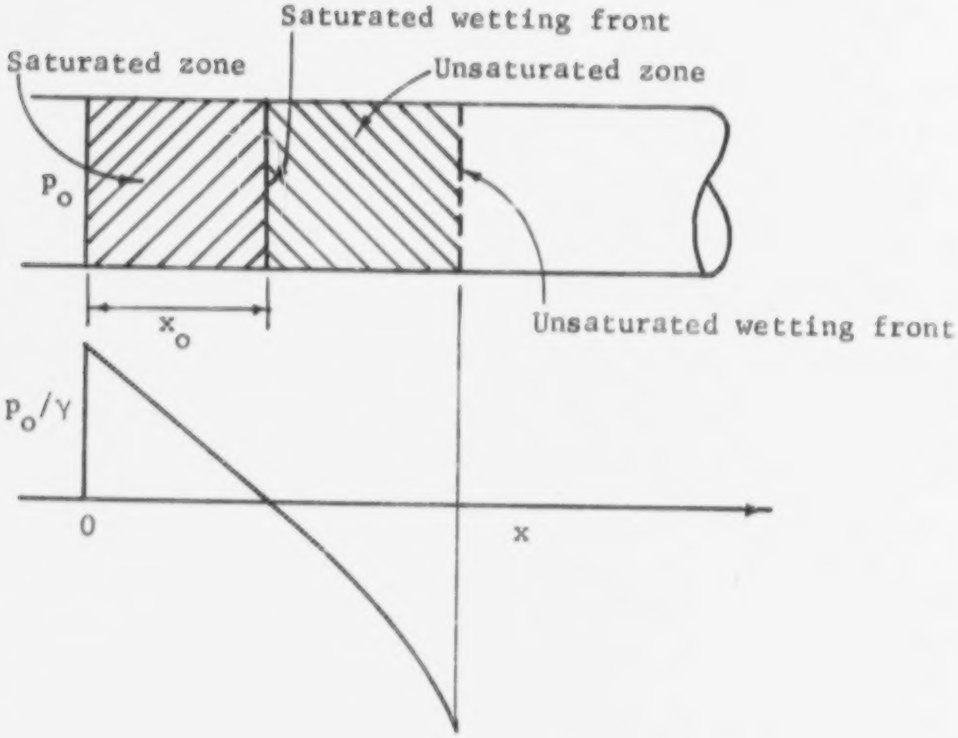

Saturated wetting front
Saturated zone
Unsaturated zone
p_o
x_o
Unsaturated wetting front
p_o/γ
0
x

Figure 5.2 Horizontal Flow into a Column of Soil Under the Influence of a Certain Applied Pressure.

The pressure drops linearly from applied pressure p_o at the surface to zero at the saturated wetting front x_o. The flow velocity, $\frac{d}{dt}\int_{\theta_n}^{\theta_s} x\, d\theta$ is related to the hydraulic energy gradient $\frac{p_o/\gamma}{x_o}$ by Darcy's law as $\frac{d}{dt}\int_{\theta_n}^{\theta_s} x\, d\theta = K_s \frac{p_o/\gamma}{x_o}$, which can be written as

$$x_o = \frac{(p_o/\gamma)\, K_s}{\frac{d}{dt}\int_{\theta_n}^{\theta_s} x\, d\theta} \qquad (5.24)$$

In the region $x > x_o$, the diffusion equation governs,

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left(D\,\frac{\partial\theta}{\partial x}\right) \qquad (5.25)$$

subject to:

$$\left.\begin{aligned} &\theta = \theta_n \quad , \quad t = 0 \quad , \quad x > 0 \\ &\theta = \theta_s \quad , \quad x = x_o \quad , \quad t \geq 0 \\ &x_o = \frac{(p_o/\gamma)\, K_s}{\frac{d}{dt}\int_{\theta_n}^{\theta_s} x\, d\theta} \end{aligned}\right\} \qquad (5.26)$$

The Boltzmann transformation,

$$\varphi = \frac{x}{(t)^{\frac{1}{2}}}$$

allows Eq. (5.25) subject to (5.26) to be reduced to a second-order nonlinear ordinary differential equation

$$\frac{d}{d\varphi}\left(D\,\frac{d\theta}{d\varphi}\right) + \frac{\varphi}{2}\,\frac{d\theta}{d\varphi} = 0 \qquad (5.27)$$

subject to:

$$\left.\begin{aligned} \theta &= \theta_n \quad , \quad \varphi = \infty \\ \theta &= \theta_s \quad , \quad \varphi = \varphi_s \\ \varphi_s &= \frac{2(p_o/\gamma)\ K_s}{\displaystyle\int_{\theta_n}^{\theta_s} \varphi \ d\theta} \end{aligned}\right\} \tag{5.28}$$

It should be noted that $\varphi_s = x_o/(t)^{\frac{1}{2}}$. Substituting $D = \alpha e^{\beta\theta}$ into Eq. (5.27) gives

$$\alpha e^{\beta\theta}\ \theta'' + \alpha\beta e^{\beta\theta}(\theta')^2 + \frac{\varphi}{2}\ \theta' = 0 \tag{5.29}$$

where

$$\theta' = \frac{d\theta}{d\varphi} \quad \text{and} \quad \theta'' = \frac{d^2\theta}{d\varphi^2}$$

Making a change in variables,

$$z = \frac{\varphi}{(\alpha)^{\frac{1}{2}}}$$

$$y = e^{\beta\theta} \quad ; \quad \theta = \frac{1}{\beta}\ \ell n\ y \quad ; \quad d\theta = \frac{1}{\beta}\ \frac{dy}{y}$$

enables Eq. (5.29) to be reduced to

$$y'' = -\ y'\ \frac{z}{2y} \tag{5.30}$$

subject to:

$$\theta = \theta_n \quad , \quad z = \infty \quad , \quad y = y_n = e^{\beta\theta_n}$$

$$\left.\begin{array}{l} \theta = \theta_s \quad , \quad z = \dfrac{\varphi_s}{(\alpha)^{\frac{1}{2}}} \quad , \quad y = y_s = e^{\beta\theta_s} \\[2ex] \varphi_s = \dfrac{2(p_o/\gamma)K_s}{\displaystyle\int_{\theta_n}^{\theta_s} \varphi \, d\theta} \end{array}\right\} \tag{5.31}$$

$$\varphi_s = \frac{2(p_o/\gamma)K_s}{\displaystyle\int_{y_n}^{y_s} z(\alpha)^{\frac{1}{2}} \frac{1}{\beta} \frac{dy}{y}}$$

$$\varphi_s = \frac{2(p_o/\gamma)K_s}{\dfrac{(\alpha)^{\frac{1}{2}}}{\beta} \displaystyle\int_{y_n}^{y_s} \frac{z}{y} \, dy}$$

Substituting $\frac{z}{y} = -2\,\frac{y''}{y'}$ into the above equation, we have

$$\varphi_s = \frac{2(p_o/\gamma)K_s}{\dfrac{(\alpha)^{\frac{1}{2}}}{\beta} \displaystyle\int_{y_n}^{y_s} \left(-2\,\frac{y''}{y'}\right) dy}$$

$$\varphi_s = \frac{-\,(p_o/\gamma)K_s\beta}{(\alpha)^{\frac{1}{2}} \displaystyle\int_{z_n}^{z_s} \left(y''\right) dz}$$

$$\varphi_s = \frac{-\,(p_o/\gamma)K_s\beta}{(\alpha)^{\frac{1}{2}} \Big[\, y' \,\Big]_{z_n}^{z_s}}$$

Finally,

$$\varphi_s = \frac{-\,(p_o/\gamma)K_s\beta}{(\alpha)^{\frac{1}{2}}\; y'(z_s)}$$

Rewriting the differential equation,

$$y'' = -\,y'\left(\frac{z}{2y}\right) \tag{5.32}$$

subject to:

$$\left.\begin{array}{lll} z = \infty & , & y = y_n = e^{\beta\theta_n} \\ z = \dfrac{\varphi_s}{(\alpha)^{\frac{1}{2}}} & , & y = y_s = e^{\beta\theta_s} \\ \varphi_s = \dfrac{-\,(p_o/\gamma)K_s\beta}{(\alpha)^{\frac{1}{2}}\; y'(z_s)} & & \end{array}\right\} \tag{5.33}$$

As before Eq. (5.33) can be written as

$$\left.\begin{array}{lll} z = z_1 & , & y = y_n = e^{\beta\theta_n} \\ z = z_s = \dfrac{-\,(p_o/\gamma)K_s\beta}{\alpha y'(z_s)} & , & y = y_s = e^{\beta\theta_s} \end{array}\right\} \tag{5.34}$$

It can be seen that Eqs. (5.21) and (5.32), subject to conditions (5.23) and (5.34), are identical. The solution of the equation will be given in the next chapter.

5.3 Effect of Non-zero Critical Suction

Because of a non-zero critical suction, some soils will exhibit a tension-saturated zone. The quantity x_o in Eq. (5.19) which was previously the distance to the saturated wetting front still represents the distance to the saturated wetting front. The only difference is that the pressure at the saturated wetting front is now negative rather than zero. Its magnitude is equal to the critical suction. Figures 1.1 and 1.2 illustrate the zero critical-suction and non-zero critical-suction conditions. For the non-zero critical-suction conditions, Eq. (5.19) becomes

$$V = K_s \frac{p_o/\gamma + S_c}{x_o} = K_s \frac{p_o/\gamma + S_c}{c(t)^{\frac{1}{2}}} \tag{5.35}$$

where S_c is the critical-suction value. Equating (5.35) and (5.18) yields

$$c = \frac{-\,(p_o/\gamma + S_c)K_s\beta}{(\alpha)^{\frac{1}{2}}\, v'(u_s)} \tag{5.36}$$

Thus, Eq. (5.23) can be written as

$$\left.\begin{aligned} u &= u_1 \qquad\qquad\qquad\qquad , \quad v = v_n = e^{\beta\theta_n} \\ u &= u_s = \frac{-(p_o/\gamma + S_c)K_s\beta}{\alpha v'(u_s)} \quad , \quad v = v_s = e^{\beta\theta_s} \end{aligned}\right\} \tag{5.37}$$

CHAPTER 6

APPROACH TO THE SOLUTION OF THE COMBINED FLOW EQUATION

6.1 Procedure

The method of solution of Eq. (5.21) subject to Eq. (5.23) is patterned after a procedure developed by Singh (1965) for unsteady unsaturated flow in two dimensions. Equation (5.21) subject to Eq. (5.23) cannot be readily solved because u_1 is unknown. This deficiency can be made up by examining the physical condition at $u = u_1$. Since

$$\theta = \theta_n \quad ; \quad \left.\frac{d\theta}{d\varphi}\right|_{\theta=\theta_n} = 0 \quad ,$$

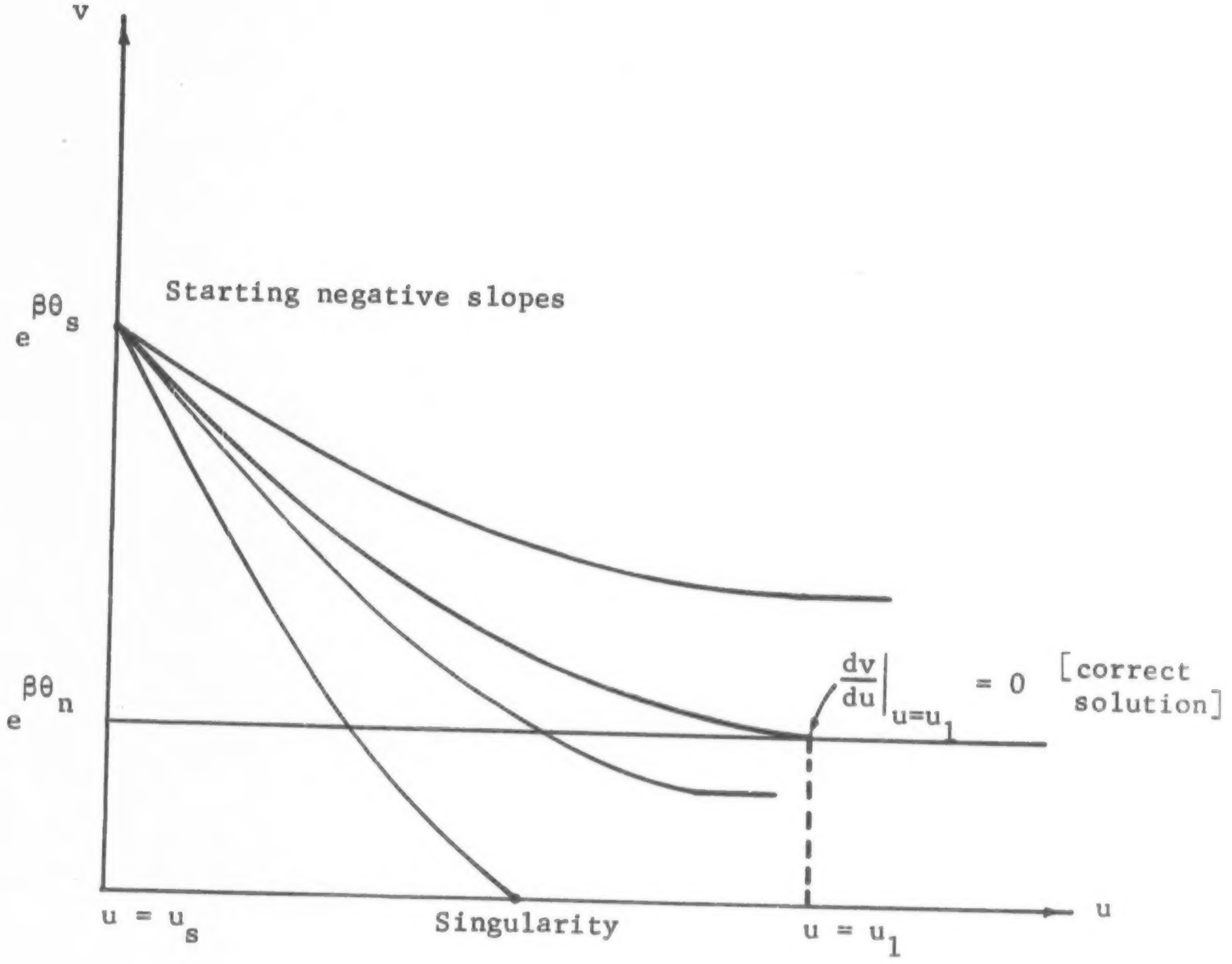
$$v = e^{\beta\theta} \quad ; \quad dv = \beta e^{\beta\theta}\, d\theta \quad ,$$

$$u = \frac{\varphi}{(\alpha)^{\frac{1}{2}}} + \frac{c}{(\alpha)^{\frac{1}{2}}} \quad ; \quad du = \frac{1}{(\alpha)^{\frac{1}{2}}}\, d\varphi \quad , \text{ and}$$

$$u = u_1 = \frac{\varphi_1}{(\alpha)^{\frac{1}{2}}} + \frac{c}{(\alpha)^{\frac{1}{2}}} \quad , \text{ therefore}$$

$$v'\,(u_1) = \left.\frac{dv}{du}\right|_{u_1} = \left.\frac{\beta e^{\beta\theta}\; d\theta}{1/(\alpha)^{\frac{1}{2}}\; d\varphi}\right|_{\theta=\theta_n} = 0$$

From Eq. (5.21), for negative v' , v'' is positive, which implies that the v curve will start curving counterclockwise as u is increased. Figure 6.1 shows the plot of v versus u for different starting negative slopes.


v
$e^{\beta\theta_s}$
Starting negative slopes
$e^{\beta\theta_n}$
$\left.\frac{dv}{du}\right|_{u=u_1} = 0$
[correct solution]
$u = u_s$
Singularity
$u = u_1$
u

$$u = \frac{\varphi}{(\alpha)^{\frac{1}{2}}} + \frac{c}{(\alpha)^{\frac{1}{2}}} \quad \text{where} \quad c = \frac{-\,(p_o/\gamma + S_c)K_s\beta}{(\alpha)^{\frac{1}{2}}\, v'(u_s)} \quad \text{and} \quad \varphi = \frac{x'}{(t)^{\frac{1}{2}}}$$

$$v = {}^{\beta\theta}$$

Figure 6.1 v-curves with Different Initial Slopes.

6.2 The Runge-Kutta Method

The combined equation for horizontal flow under a certain applied pressure has been transformed to a second order, nonlinear ordinary differential equation as given by Eq. (5.21). Its general form is

$$v'' = f(u,v,v') \tag{6.1}$$

with two given initial conditions

$$v\,(u_s) = c_1 \quad \text{and} \quad v'\,(u_s) = c_2$$

where c_1 and c_2 are two constants. Eq. (6.1) can be integrated step by step by means of the Runge-Kutta formulas, applied in the order given.

$$\begin{aligned}
k_1 &= hf(u_n, v_n, v'_n) \\
k_2 &= hf\left(u_n + \frac{h}{2}\,,\ v_n + \frac{h}{2}\,v'_n + \frac{h}{8}\,k_1\,,\ v'_n + \frac{k_1}{2}\right) \\
k_3 &= hf\left(u_n + \frac{h}{2}\,,\ v_n + \frac{h}{2}\,v'_n + \frac{h}{8}\,k_1\,,\ v'_n + \frac{k_2}{2}\right) \\
k_4 &= hf\left(u_n + h,\ v_n + hv'_n + \frac{h}{2}\,k_3\,,\ v'_n + k_3\right) \\
\Delta v &= h\left[v'_n + \frac{1}{6}\left(k_1 + k_2 + k_3\right)\right] \\
\Delta v' &= \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)
\end{aligned} \tag{6.2}$$

where $n = 0, 1, 2, \ldots$ and h is the interval between equidistant values of u. Figure 6.2 shows schematically the step-by-step integration.

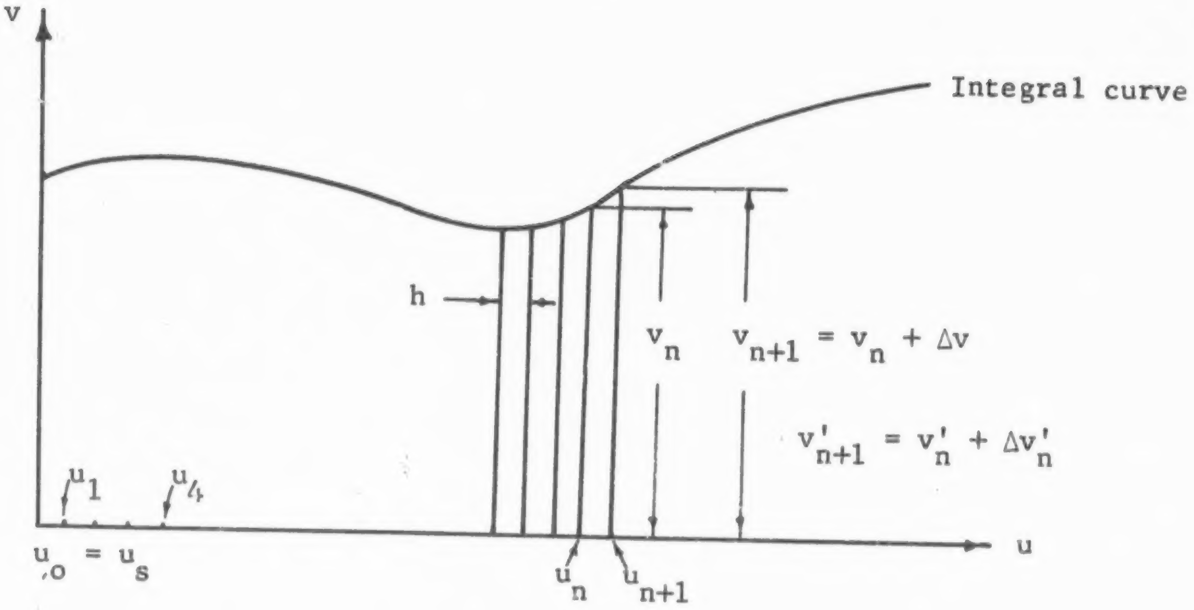

v
Integral curve
h
v_n
$v_{n+1} = v_n + \Delta v$
$v'_{n+1} = v'_n + \Delta v'_n$
u_1
u_4
$u_o = u_s$
u_n
u_{n+1}
u

Figure 6.2 Step-by-Step Integration of Runge-Kutta Method.

Equation (6.2) has been applied to solve Eq. (5.21) only for the first three steps because the accuracy of the Runge-Kutta method is limited to the 4th order; the rest of the step-by-step integrations have been carried out with the Adams predictor-corrector method using the Runge-Kutta procedure for starting.

6.3 Adams Predictor-Corrector Method

Figure 6.3 shows schematically the Adams predictor-corrector step-by-step integration of a first order differential equation of the general form $\frac{dv}{du} = f[u,v(u)]$.

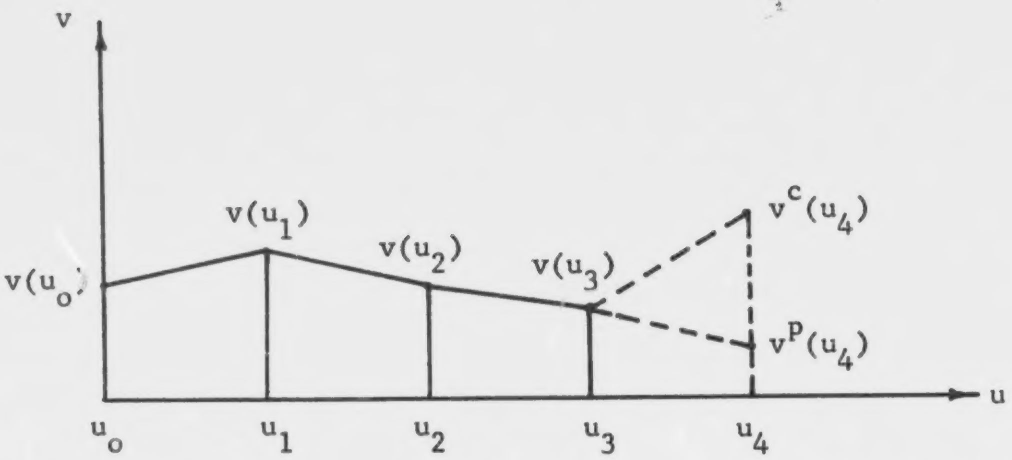


Figure 6.3 Step-by-Step Integration of Adams Predictor-corrector Method.

The Adams predictor formula is

$$v^p\ (u_4) = v\ (u_3) + \frac{h}{24}\left\{55f[u_3,v\ (u_3)] - 59[u_2,v\ (u_2)] + 37f[u_1,v\ (u_1)] - 9f[u_o,v\ (u_o)]\right\} \tag{6.3}$$

where h is the interval between equidistant values of u and the superscript p denotes the predictor. It can be seen from Eq. (6.3) that four starting values are needed. The Adams corrector formula is

$$v^c\ (u_4) = v\ (u_3) + \frac{h}{24}\left\{9f[u_4,v^p\ (u_4)] + 19f[u_3,v\ (u_3)] - 5f[u_2,v\ (u_2)] + f[u_1,v\ (u_1)]\right\} \tag{6.4}$$

where the superscript c denotes the corrector. The error bound at the

4th step is given by

$$E_{(u_4)} = \left| \frac{v^P\ (u_4) - v^c\ (u_4)}{18} \right| \tag{6.5}$$

A discussion of this method and the Runge-Kutta starting method presented above is given by Hildebrand (1956).

6.4 Method of Solution

In order to apply the preceding method to Eq. (5.21), it is necessary to replace that equation by an equivalent set of equations of the first order. Rewriting Eq. (5.21),

$$v'' = -\ v' \frac{u}{2v} \tag{5.21}$$

subject to:

$$\left. \begin{array}{ll} u = u_1 \quad , & v = v_n = e^{\beta\theta_n} \\ u = u_s = \dfrac{(-p_o/\gamma+S_c)K_s\beta}{2v'\ (u_s)} \quad , & v = v_s = e^{\beta\theta_s} \end{array} \right\} \tag{5.37}$$

The above problem is equivalent to

$$\left. \begin{array}{l} v' = g \\ g' = -\ g\left(\dfrac{u}{2v}\right) \end{array} \right\} \tag{6.6}$$

subject to:

$$v\ (u_s) = e^{\beta\theta_s}$$

$$v'\ (u_s) = g\ (u_s) = \text{assumed}$$

To solve Eq. (5.21), a trial value of $v'(u_s)$ is chosen, then u_s can be computed. Eq. (6.2) is applied three times. Thus, three more values of v corresponding to values of u are obtained; and with the initial conditions, four starting values are known. The predictor equation (6.3) is applied to Eq. (6.6) to obtain $v^P(u_4)$. The corrector equation (6.4) is applied to obtain $v^c(u_4)$. The interval h can be selected arbitrarily depending upon the accuracy required. Figure 6.4 illustrates schematically the integrating steps.

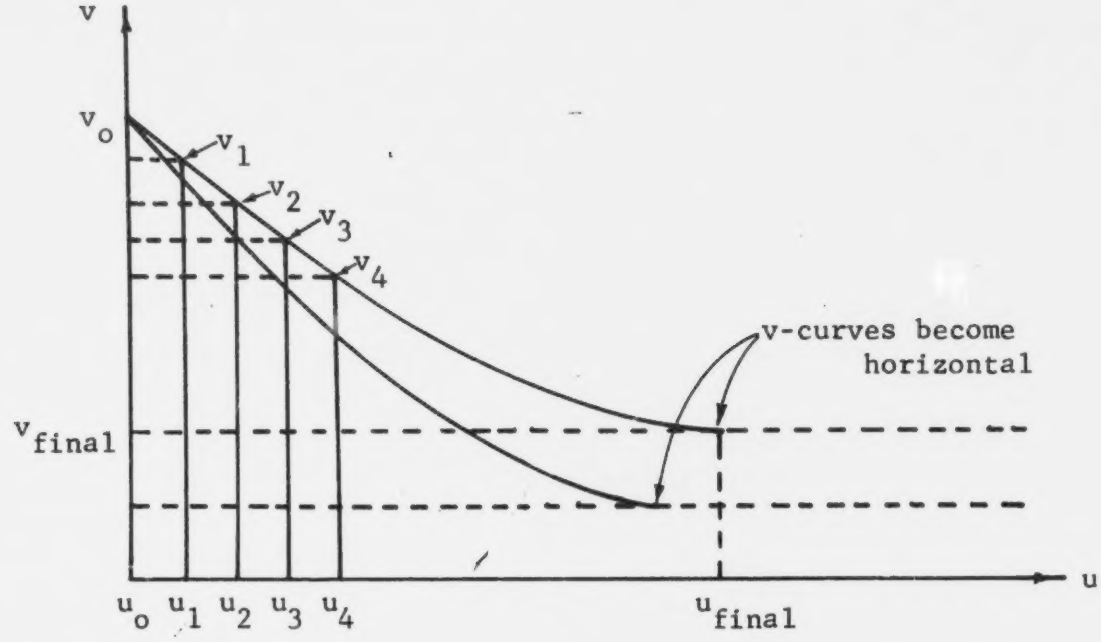


Figure 6.4 Step-by-Step Integration of a Non-linear Second-order Differential Equation.

As shown in Fig. 6.4, starting with the assumed value of $v'(u_s)$; $v(u_1)$, $v(u_2)$, $v(u_3)$, $v'(u_1)$, $v'(u_2)$, and $v'(u_3)$ are computed by the Runge-Kutta formulas, applied in the order as given in Eq. (6.2). The Adams predictor formula (6.3) is applied to Eq. (6.6) to obtain $v^P(u_4)$ and $v'^P(u_4)$. Then the Adams corrector formula (6.4) is applied to get the improved values of $v^c(u_4)$ and $v'^c(u_4)$.

The corrector formula can be applied as many times as needed. For the present work, the corrector formula is applied twice. The error is found to be in the order to 10^{-6}. The integration is carried out until the v curve becomes horizontal, which in turn implies that v' becomes zero. Then a comparison is made between $v = e^{\beta\theta_n}$ and v_{final} obtained from the solution of the differential equation. If they are not equal, a new $v'(u_s)$ is assumed and the solution of the differential equation is started all over again. Figure 6.5 shows three different v curves corresponding to three assumed values of $v'(u_s)$.

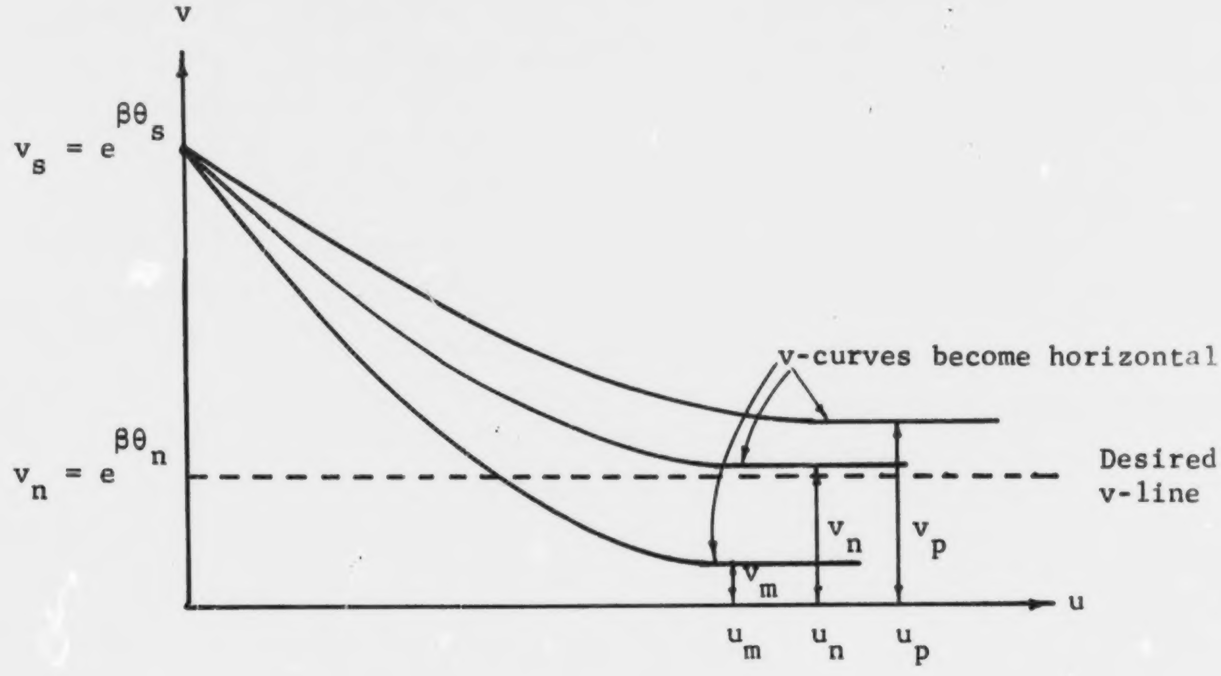

v
$v_s = e^{\beta\theta_s}$
v-curves become horizontal
$v_n = e^{\beta\theta_n}$
Desired v-line
v_n
v_p
v_m
u
u_m
u_n
u_p

Figure 6.5 v-curves with Different Assumed Initial Slopes.

If the v_{final}'s are within a reasonable range of the desired v-value, $e^{\beta\theta_n}$, the Lagrange interpolation method is used to find the $v'(u_s)$ that gives the desired v_{final} . The Lagrange interpolation evaluates at u an Nth-degree Lagrange polynomial passing through the N + 1 points (u_i, v_i) . The method is the following:

For $j = 1, 2, \ldots, N+1$, let

$$L_j\ (u) = \frac{(u-u_1)\ \ldots\ (u-u_{j-1})(u-u_{j+1})\ \ldots\ (u-u_{N+1})}{(u_j-u_1)\ \ldots\ (u_j-u_{j-1})(u_j-u_{j+1})\ \ldots\ (u_j-u_{N+1})} \tag{6.7}$$

Then $L_j\ (u)$ is a polynomial of degree N with a value of unity when $u = u_j$ and 0 when $u = u_i$, $i \neq j$. The polynomial $L_j\ (u)\ v_j$ then has the value v_j at $u = u_j$, zero, otherwise. Therefore,

$$v\ (u) = \sum_{j=1}^{N+1} L_j\ (u)\ v_j \tag{6.8}$$

is a polynomial of degree N passing through the $N+1$ points (u_i, v_i) . Three values of v_{final}'s obtained from the solution of the differential equation are used as the independent variable and the corresponding $v'\ (u_s)$'s are used as the dependent variables. Equations (6.7) and (6.8) are applied to obtain an interpolated value of $v'\ (u_s)$. For this case, the Lagrange polynomial is of second degree. Using the interpolated $v'\ (u_s)$'s , the differential equation is re-solved and a comparison is made again between the v_{final} and $v_{desired}$. If they are not equal, Eqs. (6.7) and (6.8) are applied again but with four independent and dependent variables. The method can be applied successively until a desired value of v is obtained within the tolerance of accuracy.

Once the solution of the differential equation is achieved, the following equations are used to obtain the moisture profile as a function of time and distance:

$$\left.\begin{aligned} \theta &= \frac{1}{\beta}\ \ell n\ v \\ x' &= (t)^{\frac{1}{2}}[u(\alpha)^{\frac{1}{2}} - c] \\ x_o &= c(t)^{\frac{1}{2}} \end{aligned}\right\} \tag{6.9}$$

where

$$c = \frac{-\,(p_o/\gamma + S_c)K_s\beta}{(\alpha)^{\frac{1}{2}}\ v'\ (u_s)}$$

6.5 Operation of the Computer Program

The above mentioned method of solution of Eq. (5.21) subject to (5.37) has been programmed for use on a digital computer. The aspects of the ALGOL program are presented in Appendix I. In order to use the program, one must know the physical characteristics of the soil: K_s , $D(\theta)$, S_c , θ_n , and θ_s . The program applies only to soils having an exponential soil moisture diffusivity. Table 6.1 outlines the required input data and changes to be made in the program for different soils.

TABLE 6.1

OPERATION OF THE COMPUTER PROGRAM

Card No.	Identifiers	Type I, Integer R, Real	Comments
1,2,3,4,5			For information only
18	REAL ARRAY	I	If the number of integration steps is greater than 999, the upper bound of arrays X, Y, and DY should be increased as desired.
35	KSAT	R	Value of K_s to be assigned.
36	PRE	R	Value of applied pressure to be assigned. If soil exhibits a non-zero critical suction, then this value should be added to the applied pressure.
37	AL	R	Value of α to be assigned.
38	BET	R	Value of β to be assigned.
39	TMX	R	Value of θ_s to be assigned.
40	TMN	R	Value of θ_n to be assigned.
41	DY[0]	R	First assumed value of initial slope.
42	G[1]	R	Second assumed value of initial slope.
43	G[2]	R	Third assumed value of initial slope.

Card No.	Identifiers	Type	Comments
44	II	I	Total integration steps (can be changed if necessary)
45	H	I	Integration interval (can be changed if necessary)
61	IF STATEMENT	I	Minimum interval of steps, at which output of step by step integration is desired. Should divide II evenly.
64,65,66			For information only.
86	IF STATEMENT	I	Same as card no. 61.
92	II	I	A convenient number by which II should be increased, in case integration has not become asymptotic as yet.
97			Tolerance, can be changed as desired.
102			For information only.
113			For information only.
114	MOIST	R	θ_s should be assigned to MOIST [I:0]
115	FOR STATEMENT	I	Initial value of I and its increment can be selected to get as many as points on the moisture profile as desired, but must divide II evenly.
122	P	I	$P = II/I_{initial}$.
125	FOR STATEMENT		Initial value of I and its increment can be selected to get as many output values as a function of time as desired. I values correspond to different times in this for-loop statement.
130	FOR STATEMENT		Same as card no. 115.
135			For information only.

Table 6.2 illustrates the output identifiers and their actual terms.

TABLE 6.2

IDENTIFIERS IN OUTPUT

Identifiers in Output	Actual Term
DY	$v'(u_s)$
PHI	$c = \frac{-(p_o/\gamma + S_c)K_s\beta}{(\alpha)^{\frac{1}{2}}\, v'(u_s)}$
X	Correspond to u
Y	Correspond to v
MAX ERROR	Maximum error in integration
T	Time
X	Horizontal distance from source
THETA	θ
SATURATION	Degree of saturation = θ/θ_s
IVOL	Accumulative absorption
IRAT	Flow rate
XSAT	Saturated wetting front

As a check on the developed theory and computer program, an example for Pachappa loam (Gardner and Mayhugh, 1958) was chosen. The hydraulic characteristics for this soil are as follows:

$$\theta_s = 0.365$$

$$\theta_n = 0.06$$

$$D = 9.1 \times 10^{-3} e^{19.7\theta}$$

$$D(\theta_n) = 3 \times 10^{-2} \text{ cm}^2/\text{min.}$$

$$\frac{D(\theta_s)}{D(\theta_n)} = 400$$

Figure 6.6 shows the moisture content distribution at three different times. The solid lines represent solutions of the differential equation by the above mentioned numerical method. The circles are experimental data by Gardner and Mayhugh.

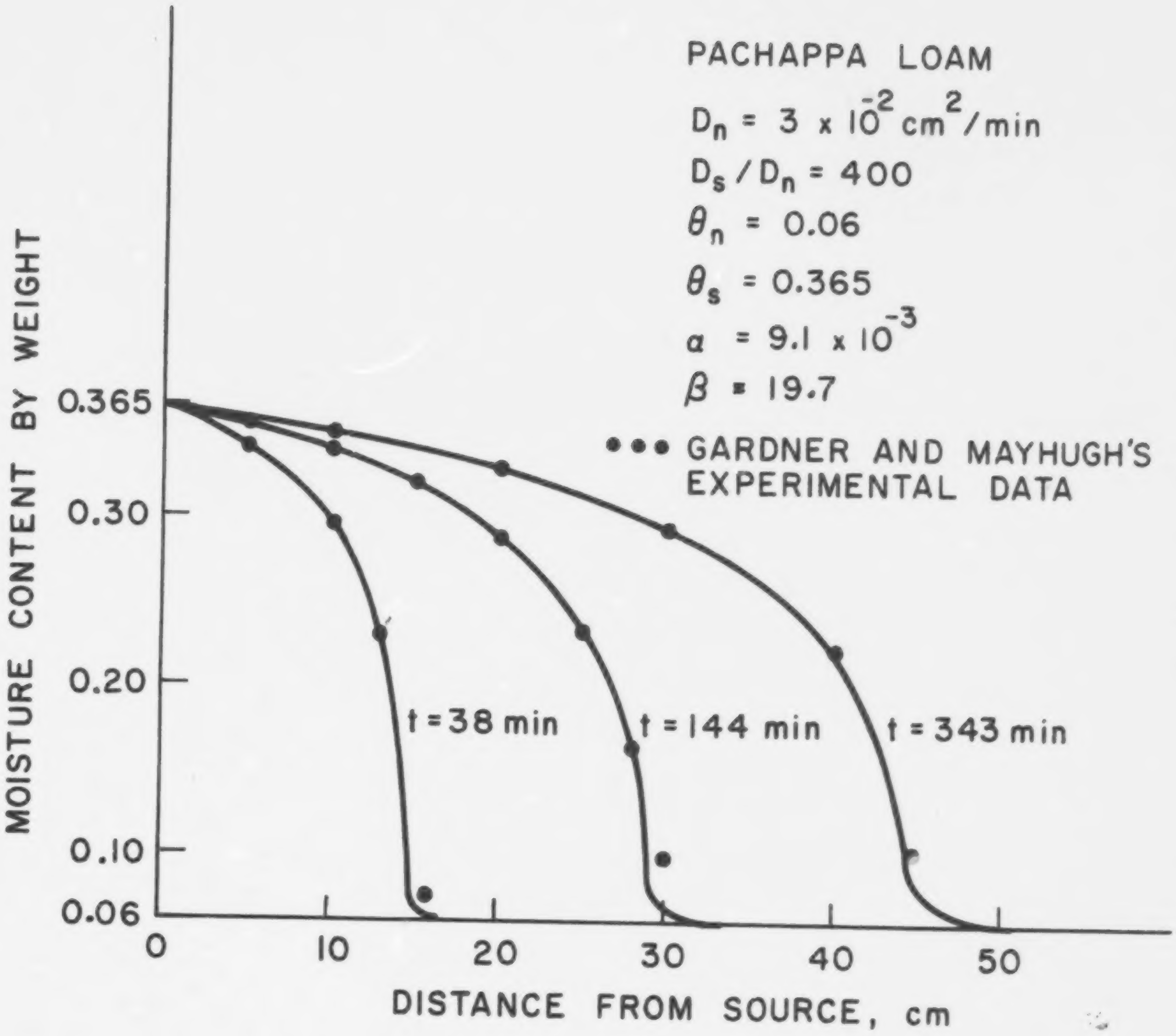


Figure 6.6 Moisture Distribution for Pachappa Loam (After Gardner and Mayhugh, 1958).

Computations were also made for several cases of silica flour. These corresponded to the different applied pressures used in the experiments. The plotted results are shown in Figs. 7.21, 7.22, and 7.23. Appendix I shows the computer program and a typical output for silica flour under the influence of a certain applied pressure.

CHAPTER 7

EXPERIMENTS

7.1 Description of Experimental Apparatus

The rectangularly-shaped plastic tube into which the soil was packed was made of premium preshrunk plastic some 5 feet long with inside dimensions of 6.35 cm width by 4.445 cm height. The γ-rays were directed through the width of 4.445 cm. The plastic was 5/8-inch in thickness to a tolerance of ± 0.005 inch. The plastic tube, similar to the one used by Adrian (1964), was assembled with machine screws. A rectangular aluminum channel about six feet long with steel rods on sides and bottom was bolted to the top of a large wooden table. The inside dimension of the channel fitted exactly the outside dimension of the plastic tube when the lower one-third portion of the plastic tube was protected by a galvanized iron sheet. Figure 7.1 shows a cross-sectional view of the plastic tube resting in the aluminum channel.

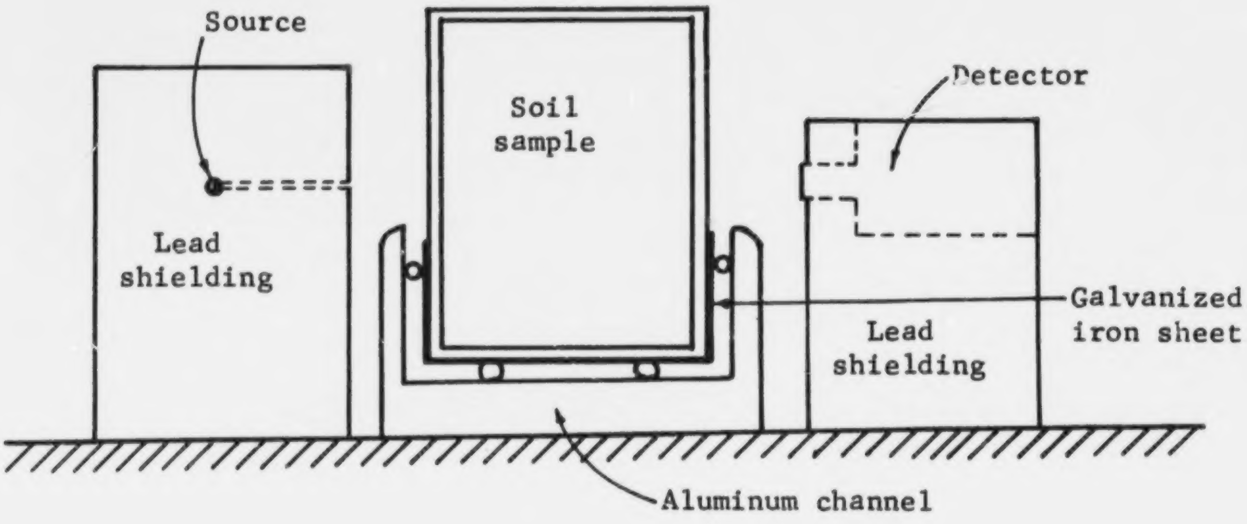


Figure 7.1 A Cross-sectional View of the Plastic Tube Resting in the Aluminum Channel.

A cloth tape was attached on top of the plastic tube. The plastic tube could slide horizontally along the channel from one position to another with little effort to overcome the frictional resistance. A point gage was aligned in the same direction as the collimated γ-ray beam to locate the position of the soil sample relative to the beam. Water was supplied from a large cylindrical pressure tank. The tank was two-thirds full of water and one-third full of compressed air. The water pressure was controlled by adjusting the air pressure in the tank. Figure 7.2 illustrates the flow system and pressure control device.

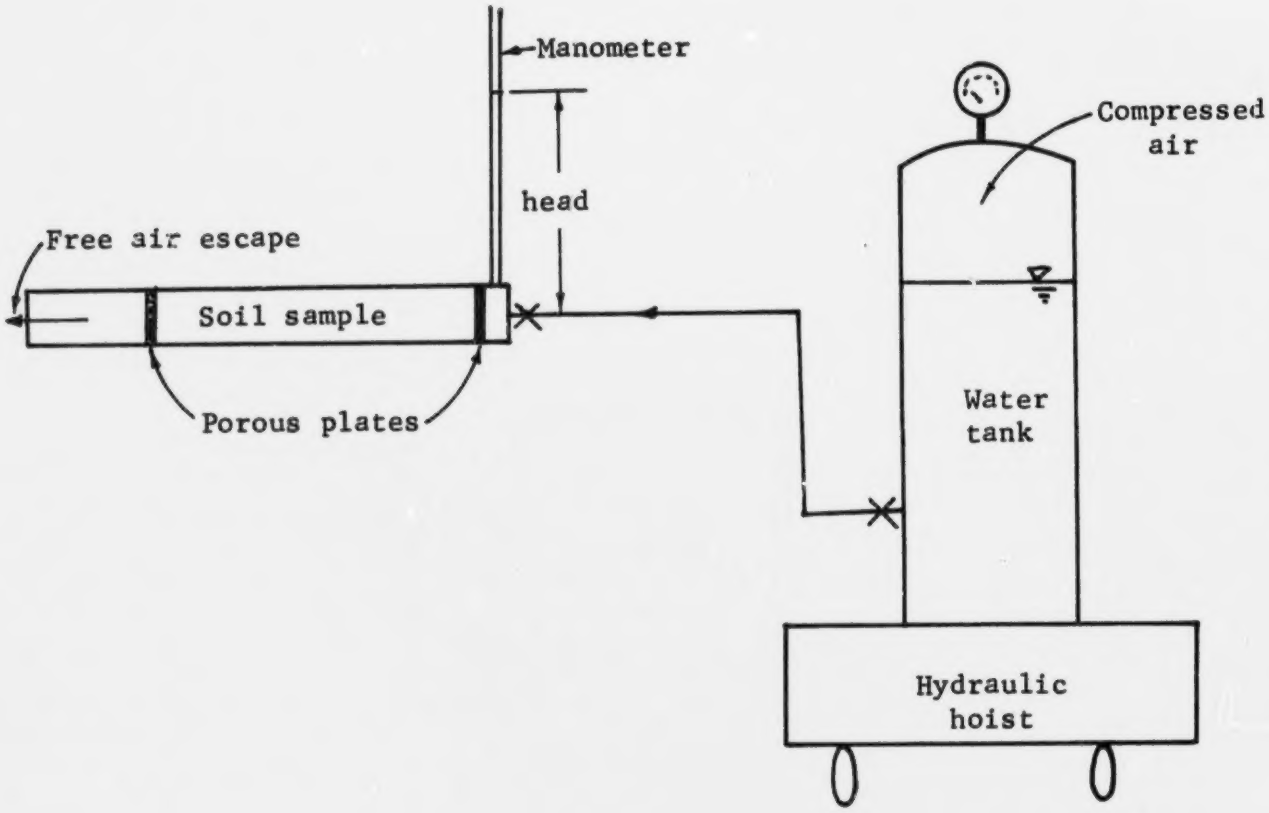
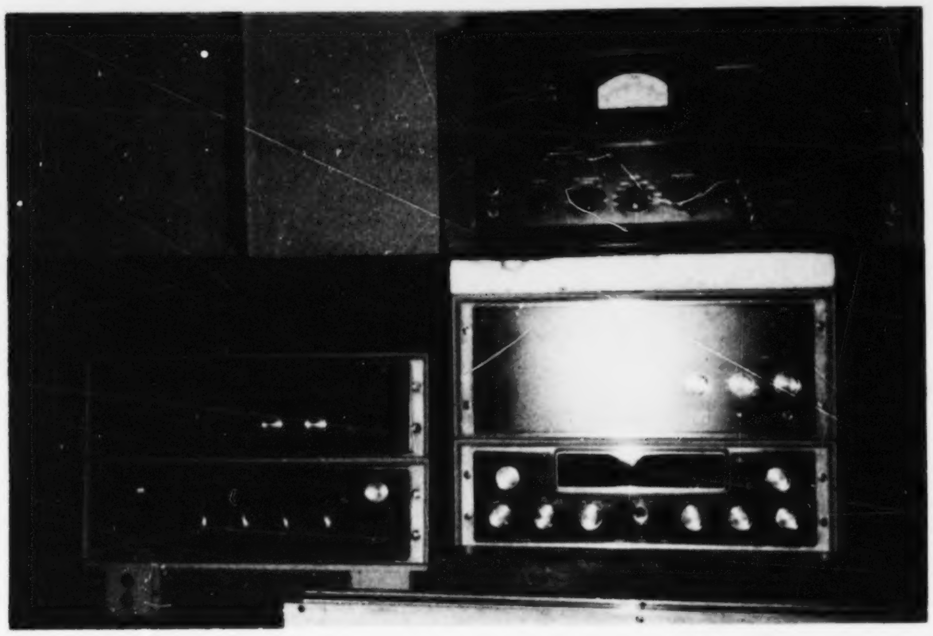

Manometer
Compressed air
head
Free air escape
Soil sample
Porous plates
Water tank
Hydraulic hoist

Figure 7.2 Schematic Diagram of Flow System.

The diameter of the tank was considerably larger than the cross-sectional dimensions of the plastic tube. Thus, constant head could be assumed over short periods during a run. The water tank was raised intermittently to maintain a constant head.

Photographs of the apparatus are shown in Figs. 7.3, 7.4, 7.5, 7.6, 7.7, and 7.8.



LINEAR AMPLIFIER
PULSE HEIGHT ANALYZER
COOLING BASE

RATEMETER
HIGH VOLTAGE SUPPLY
SCALER

Figure 7.3 Gamma-ray Attenuation Equipment.

Figure 7.4 Voltage Stabilizer and Powerstat Voltage Regulator.

NOT REPRODUCIBLE

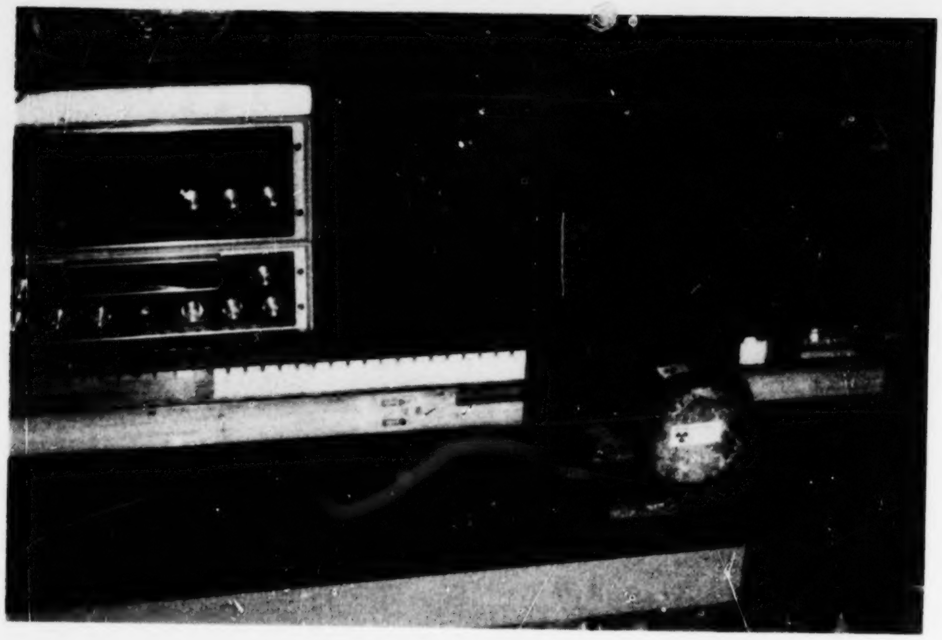


Figure 7.5 Photograph of Plastic Tube.

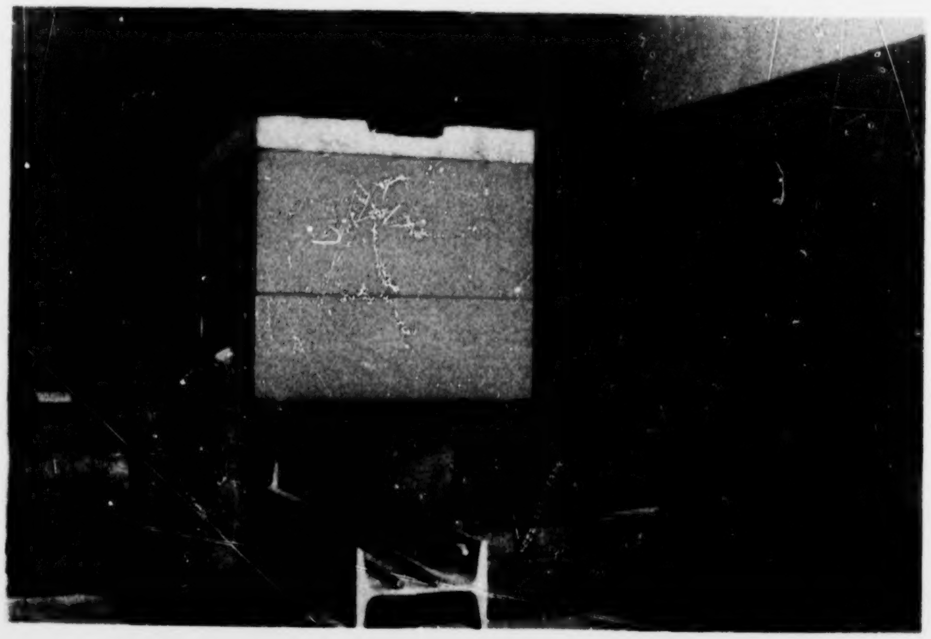


Figure 7.6 Photograph of Aluminum Channel.

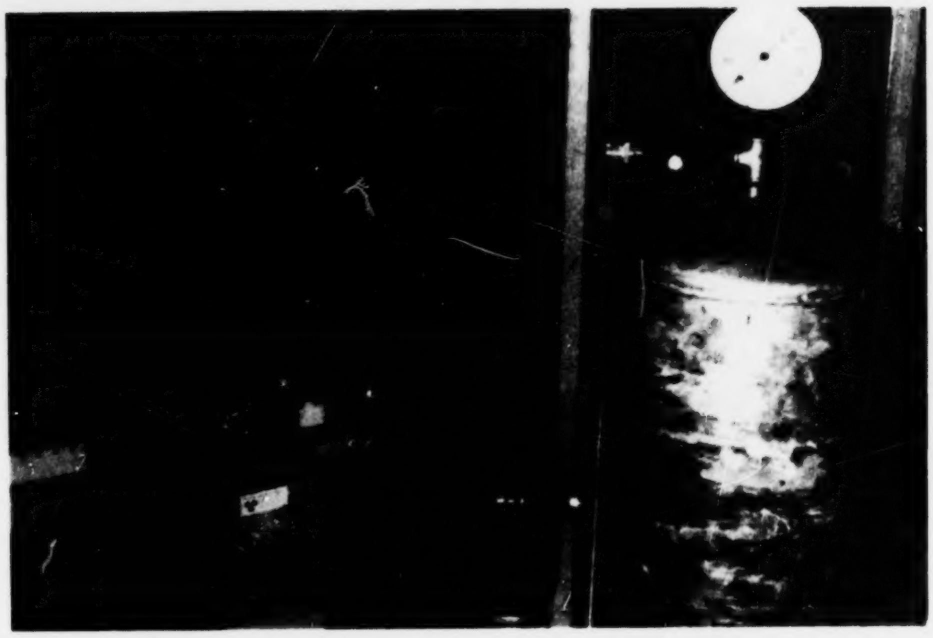


Figure 7.7 Photograph of Water Supply, Gamma-ray Source, and Scintillation Detector.

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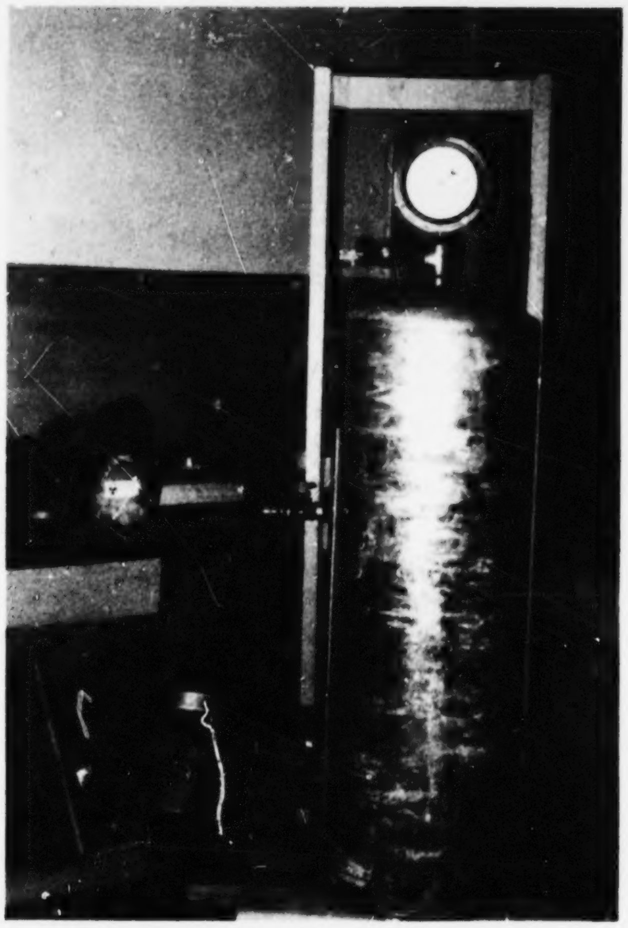

Figure 7.8 Photograph of Pressurized Water Tank.

7.2 Description of the Porous Medium

The porous medium used in the horizontal flow test was silica flour. A commerical product marketed by the Ottawa Silica Company was used. The material is called #395 SIL CO SIL. According to the manufacturer, 98% by weight of this material passes through U.S. Sieve No. 325. The specific gravity of the silica flour is 2.65.

7.3 Soil Sample Packing Procedure

Due to the rectangular shape of the plastic tube, it was impossible to utilize any of the mechanized packers that have been developed. To prevent water from forming channels or changing the structure of the soil sample when under pressure, a suitable packing procedure was necessary. The plastic tube was held vertically by one person. A second person poured the soil material uniformly through a funnel connected to a long aluminum tube, while a third person was striking the plastic tube with his hands. Thus, segregation of the particles by free fall was prevented. Each time a 4-inch layer of soil was formed in the plastic tube; the soil was compacted with twenty-five evenly distributed blows from a 3.5-lb rectangular rod, using a one-foot drop. The procedure was repeated up to any desired length of column. Then the γ-ray equipment was used to check the uniformity of the packing. If necessary, the plastic tube could be repacked until uniform bulk density was achieved.

7.4 Gamma-ray Attenuation Equipment

The transient moisture content at different positions was measured by use of γ-ray attenuation equipment. The γ-ray attenuation equation for a fixed source-detector distance is as follows:

$$I_w = I_o \exp[-(\mu_s \rho_s x_s + \mu_p \rho_p x_p + \mu_w \rho_w x_w)] \qquad (7.1)$$

where

I_o = radiation intensity with no interference

I_w = intensity of the transmitted radiation through the wet tube

μ_s, μ_p, μ_w = mass absorption coefficients for the soil, plastic, and water respectively (cm^2/g)

ρ_s, ρ_p, ρ_w = densities of soil, plastic, and water respectively (g/cm^3), and

x_s, x_p, x_w = total thickness of soil, plastic or water traversed by the transmitted γ-ray beams (cm).

When the tube is dry, Eq. (7.1) becomes

$$I_d = I_o \exp[-(\mu_s \rho_s x_s + \mu_p \rho_p x_p)] \tag{7.2}$$

where I_d is the intensity of the transmitted radiation through the dry tube. Dividing Eq. (7.1) by Eq. (7.2) yields

$$\frac{I_w}{I_d} = \exp[-\mu_w \rho_w x_w] \tag{7.3}$$

The product $\mu\rho$ is defined as the linear attenuation coefficient. Equation (7.3) may be written as

$$\frac{I_w}{I_d} = \exp[-\sigma_w x_w] \tag{7.4}$$

where σ_w is the linear attenuation coefficient for water. It is more convenient to rearrange the attenuation equation to express the moisture content, θ, of the soil directly as a function of the linear attenuation coefficient. This is achieved by multiplying and dividing by ℓ/ρ_w in the exponent of Eq. (7.4) to obtain

$$\frac{I_w}{I_d} = \exp\left[\left(-\frac{\sigma_w \ell}{\rho_w}\right)\left(\frac{x_w \rho_w}{\ell}\right)\right] \tag{7.5}$$

where ℓ is the distance between the walls of the plastic tube traversed by γ-ray beams. The quantity $x_w \rho_w / \ell$ is the moisture content of soil (θ) and has the dimension of g/cm^3. It is numerically the same whether expressed in g/cm^3 or cm^3/cm^3 for water. Equation (7.5) can be written as

$$\frac{I_w}{I_d} = \exp\left[-\left(\frac{\sigma_w \ell}{\rho_w}\right)(\theta)\right]$$

Solving for θ yields

$$\theta = -\frac{\rho_w}{\sigma_w \ell} \ln \frac{I_w}{I_d} \tag{7.6}$$

The set-up of the γ-ray attenuation equipment is shown schematically in Fig. 7.9.

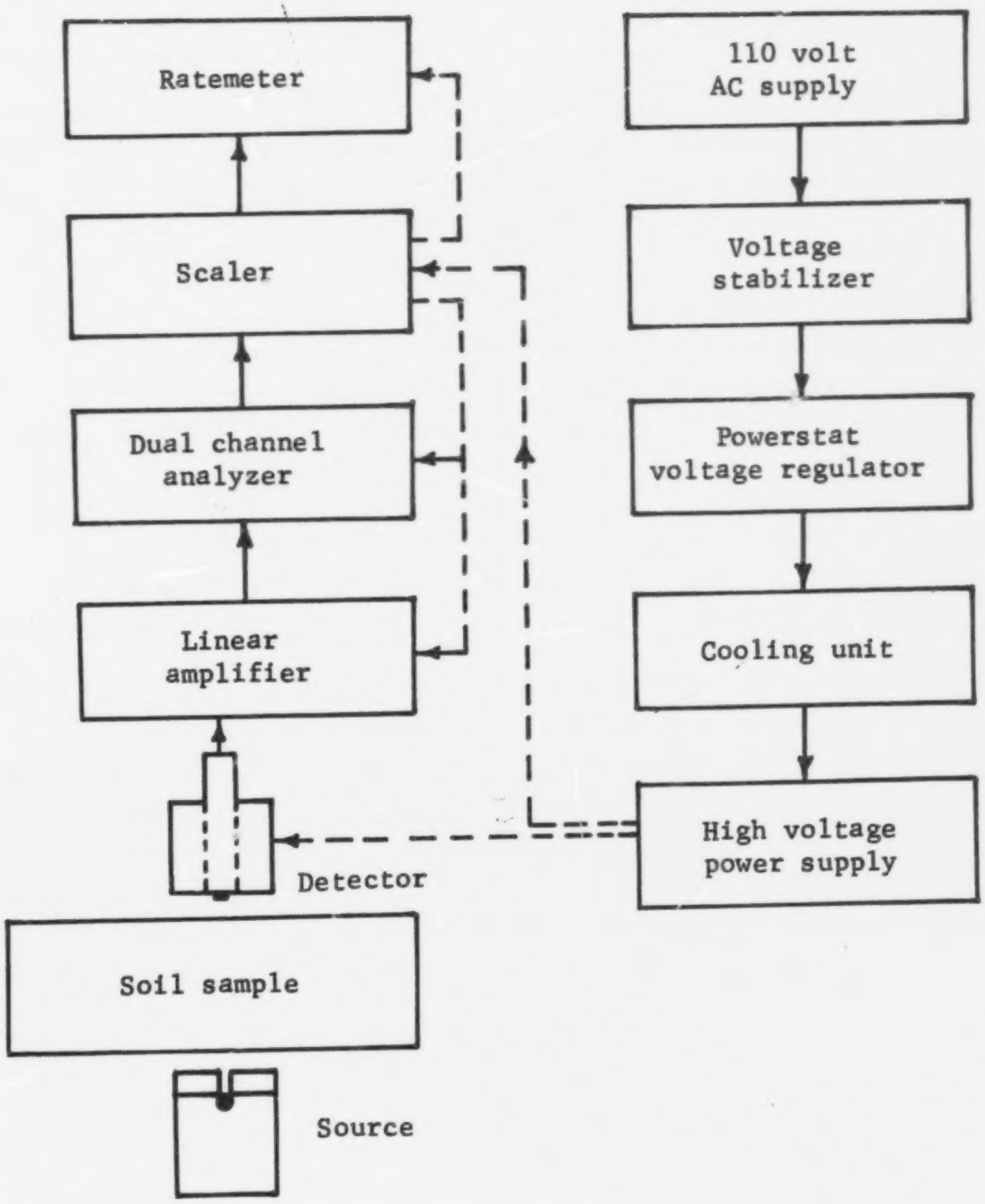
Ratemeter
110 volt AC supply
Scaler
Voltage stabilizer
Dual channel analyzer
Powerstat voltage regulator
Linear amplifier
Cooling unit
High voltage power supply
Detector
Soil sample
Source

Figure 7.9 Schematic Diagram of γ-ray Attenuation Equipment for Measuring the Moisture Content of Soil in a Laboratory.

The voltage stabilizer and powerstat voltage regulator provide a constant steady voltage supply to the system. The source of γ-ray was a 250 millicurie, hermetically-sealed source of C_s^{137}. A large part of the emitted radiation has an energy of 0.662 Mev. The γ-rays were collimated through a ¼-inch slit in the lead shielding followed by a 3/32-inch diameter slit. After travelling through the tube, the γ-rays were received at the nose of the scintillation detector. The voltage for the detector was supplied by a high voltage component. The detector output was directed to a linear amplifier, then the signals were sorted in the dual channel analyzer. The base line and window opening on the pulse height analyzer were set to accept γ-rays of 0.662 ± 0.025 Mev. The scaler recorded the pulses accepted during any preset counting period from 0.1 to 100 minutes. In order to get an instantaneous counting rate, a ratemeter was connected to the scalar. The time constant selected for the ratemeter varied from 2.5 sec, 10 sec. to 40 sec, depending on how fast the moisture was changing. For the purpose of checking on the operation of the equipment, the mass absorption coefficients of several materials of known thickness were determined. The procedure in determining the mass absorption coefficients of a material consisted in placing different thicknesses of material between the source and scintillation detector and measuring the transmitted radiation intensity over a 4-minute counting period. As predicted by Eq. (7.1), the resulting plot of relative transmittance versus thickness of material yields a straight line as shown in Fig. 7.10. Table 7.1 shows the mass absorption coefficients for several materials.

TABLE 7.1

Mass Absorption and Linear Attenuation Coefficients

Material	Linear Attenuation Coefficients, cm^{-1}	Densities, gm/cm^3	Mass Absorption Coefficients, cm^2/gm
Lead	1.1759	11.3431	0.1037
Aluminum	0.1965	2.702	0.0727
Plastic	0.0987		
Water	0.0823	0.9973	0.0825

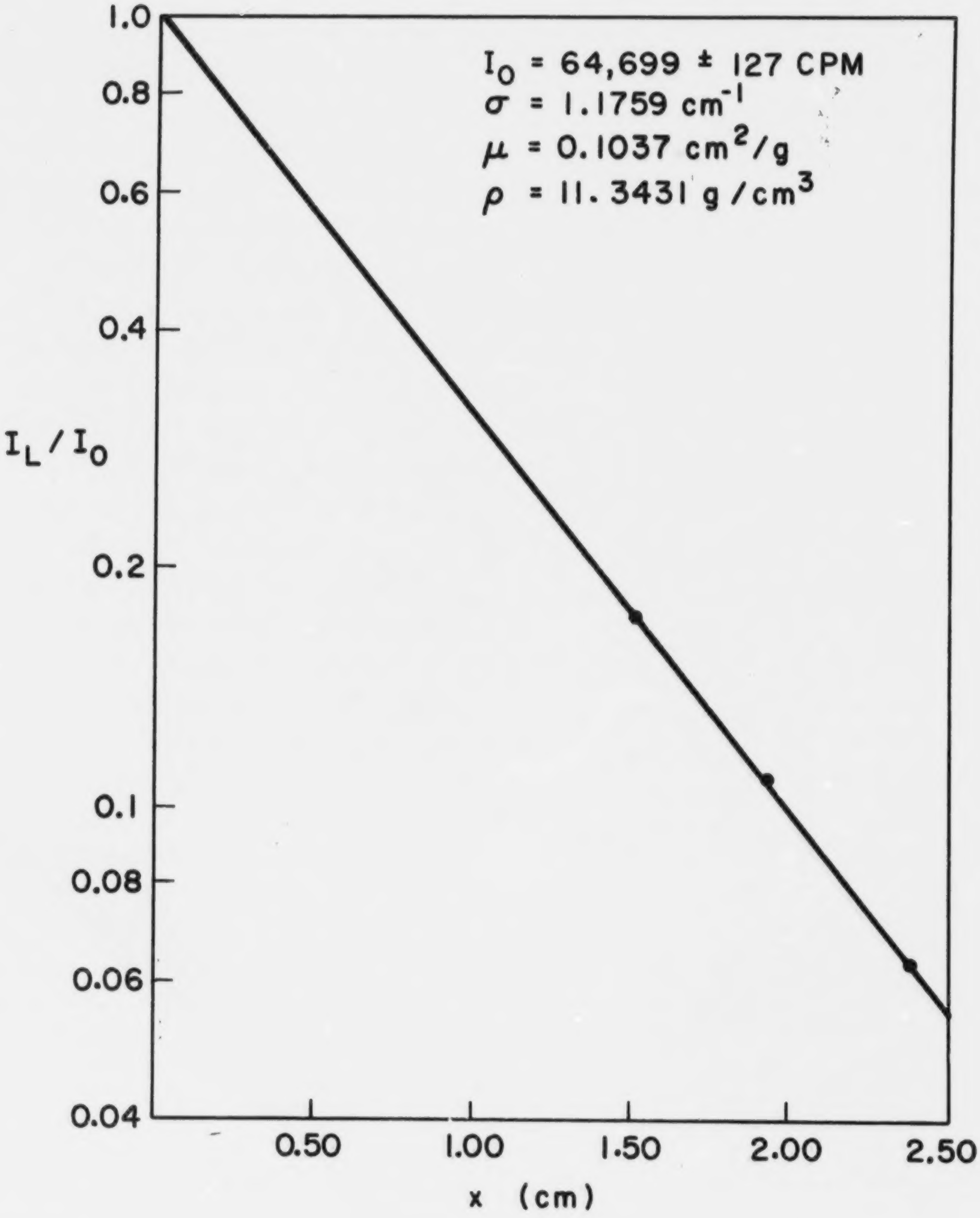

Figure 7.10 Linear Attenuation Coefficient for Lead.

7.5 Moisture Profile During Horizontal Flow with Zero Applied Pressure

As shown in Fig. 7.2, the pressurized water tank permitted the supply of water at constant head. The applied pressure head was recorded from the manometer. The decrease in head due to water absorption into the column could be corrected by either increasing the air pressure or raising the water tank. The determination of the moisture content as a function of position and time was accomplished as follows. The stopwatch was started at the instant contact was established between the wetted porous stone and the soil sample. Measurement of the γ-ray beam at any one location along the tube was recorded over a short time period, the tube being moved successively from one position to another. To obtain an instantaneous moisture profile, readings were taken along the tube at successive locations in the flow direction up to the visible wetting front, and then at successive locations in the opposite direction. Measurements were taken at the same locations in both directions and the average of the two readings was taken as the one applicable for the time at which the repositioning of the successive readings was reversed in direction. By use of Eq. (7.6) measurements of the counting rate before wetting and during wetting permitted direct calculation of moisture content. Figure 7.11 illustrates the moisture distribution curves for zero applied pressure. The circles are experimental data.

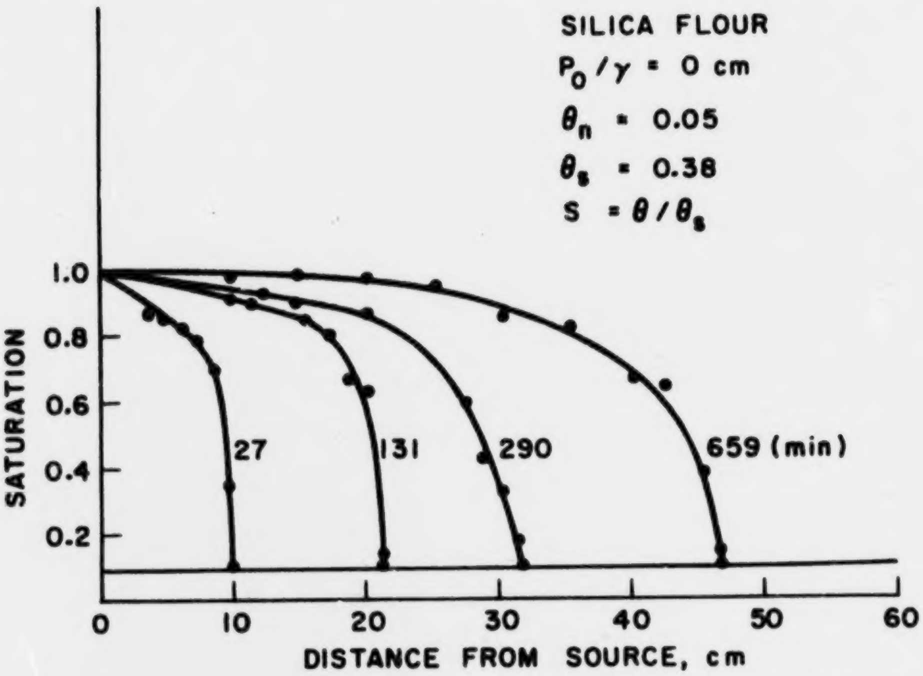

Figure 7.11 Experimental Moisture-content Distribution Curves Faired through Data Points (p_o/γ = 0 cm).

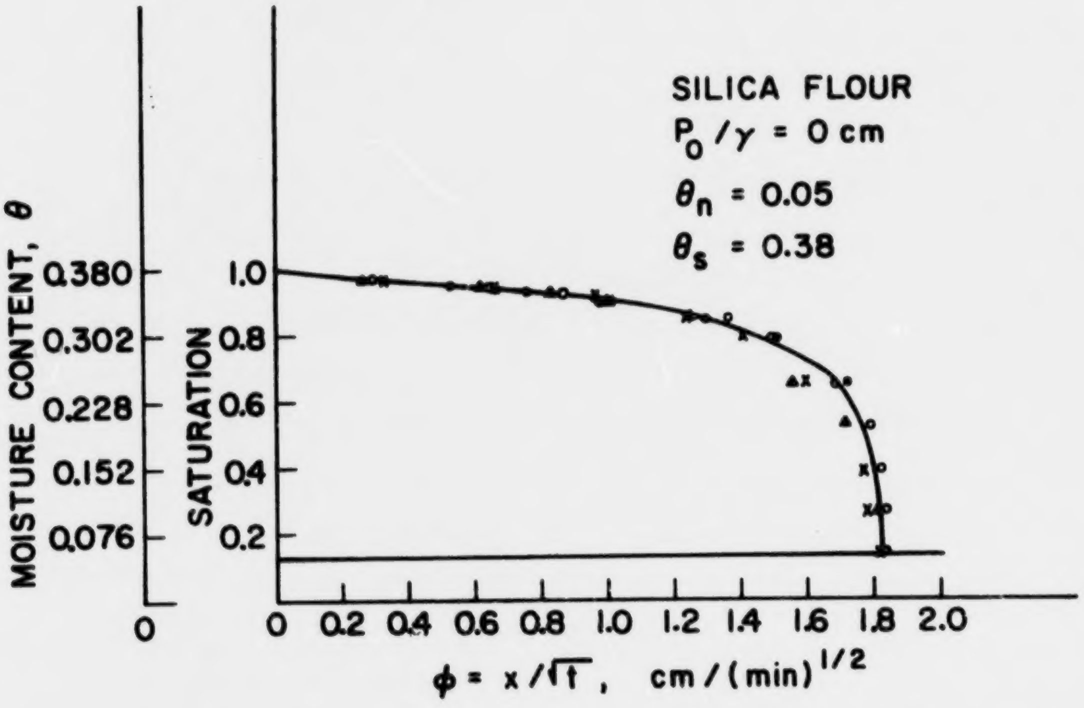

Figure 7.12 Water-content Distribution Curve for Silica Flour.

7.6 Determination of Soil Moisture Diffusivity

From Eq. (4.3), we have

$$\frac{d}{d\varphi}\left(D\,\frac{d\theta}{d\varphi}\right) + \frac{\varphi}{2}\,\frac{d\theta}{d\varphi} = 0 \qquad (4.3)$$

subject to the conditions:

$$\theta = \theta_s \quad , \quad \varphi = 0 \quad ; \quad \theta = \theta_n \quad , \quad \varphi = \infty$$

Integrating leads to

$$\int_{\theta_n}^{\theta} \varphi\, d\theta = -\,2D\,\frac{d\theta}{d\varphi} \qquad (7.7)$$

The lower limit of the integral is fixed as θ_n and Eq. (7.7) is subject to $\theta = \theta_s$, $\varphi = 0$. Solving for D in Eq. (7.7) yields

$$D(\theta) = -\,\frac{1}{2}\,\frac{d\varphi}{d\theta}\int_{\theta_n}^{\theta} \varphi\, d\theta$$

The slope $\frac{d\varphi}{d\theta}$ and the integral can be evaluated at a particular value of θ . Figure 7.12 shows a plot of moisture content, θ , versus $x/(t)^{\frac{1}{2}}$. As can be seen from this plot, accurate determination of diffusivity cannot be made at the low water contents nor at water contents near full saturation. This is due to the fact that the slope of the curve approaches zero at the low water contents and becomes very large near full saturation. As predicted by the empirical equation, i.e., $D = \alpha e^{\beta\theta}$, the plot of D versus θ on a semi-logarithmic paper yields a straight line as shown in Fig. 7.13. The empirical constants α and β for the silica flour were found to be 7.7×10^{-4} and 25. Hence, from this procedure it is concluded that the diffusivity of the silica flour used in the tests can be expressed by $D = 7.7 \times 10^{-4} e^{25\theta}$ cm^2/min .

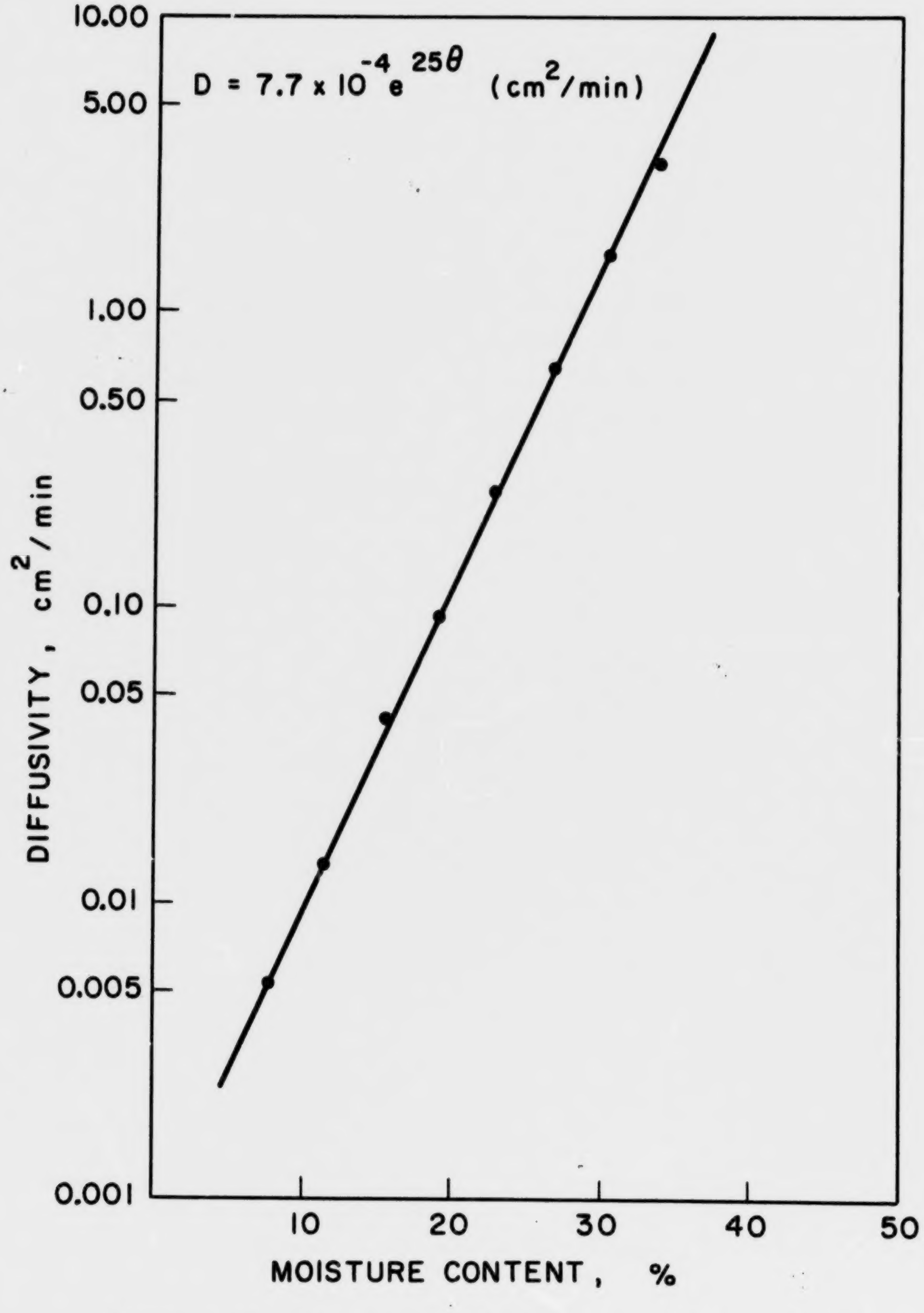

Figure 7.13 Soil-water Diffusivity vs. Moisture Content for Silica Flour.

7.7 Experimental Determination of the Saturated Hydraulic Conductivity

A permeability test was conducted on the silica flour medium using a simple upward-flow permeameter. Special effort was made to reproduce the same packing used in the experiments. The saturated hydraulic conductivity was found to be 1.33 x 10^{-3} cm/min.

7.8 Experimental Determination of the Critical-suction Value

Figure 7.14 shows the apparatus that may be used to determine approximate values of critical-suction. It consists of a porous-plate funnel connected to a burette by means of flexible tubing. To remove the

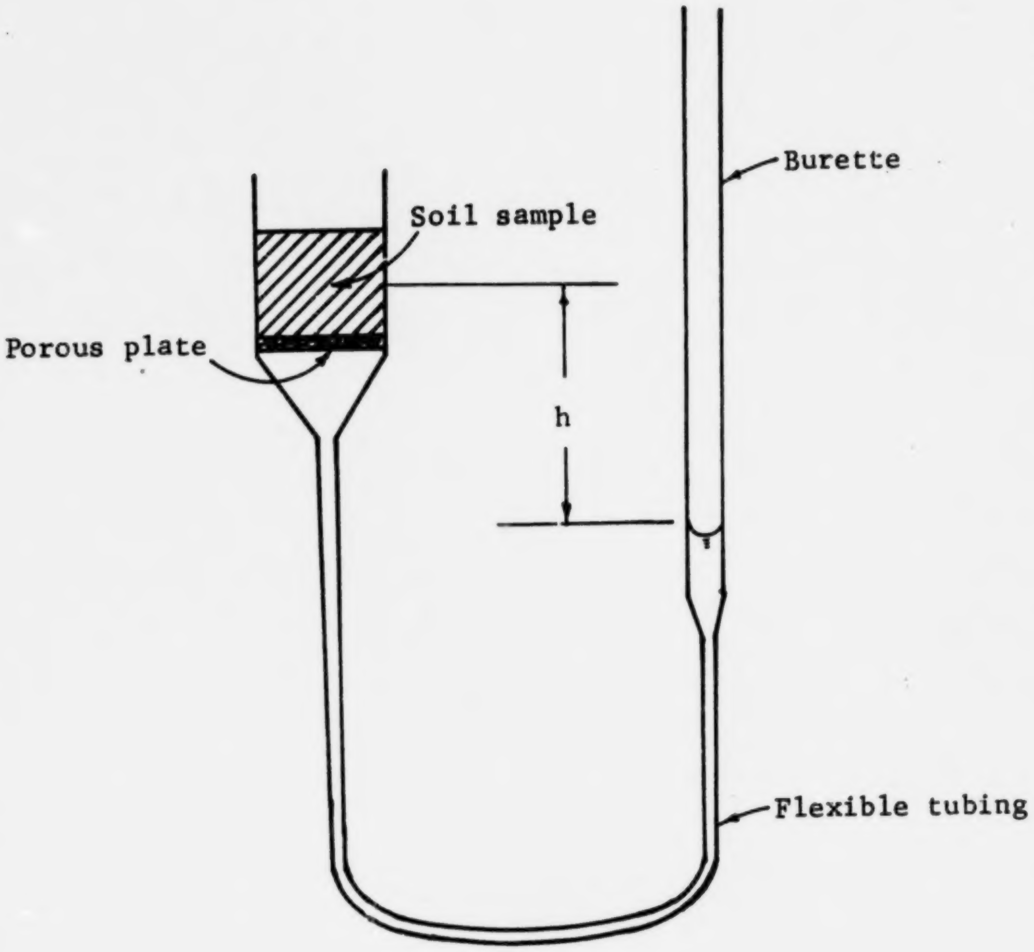

Figure 7.14 Schematic Diagram of Apparatus for Critical-suction Determination.

air from the porous plate and from the space below the plate including the tubing leading to the burette, an upward flow is introduced by filling the burette with water while the soil sample is held in place by putting another porous plate on top of it. The flow is continued until water appears on the top porous plate. Then the top porous plate is removed. The quantity of water in the system is adjusted so that the water in the burette is level with the top edge of the soil sample. Excess water on top of the sample is siphoned out. To determine the critical-suction value of the sample the procedure is as follows. The funnel is raised a certain small distance h from the surface of the water in the burette. This distance is recorded and an observation is made to note if there is any removal of water from the soil sample. This procedure is repeated successively with an interval of about fifteen minutes between successive positionings of the funnel. Finally a sudden drainage will occur if the soil has a clear-cut value of critical suction. The h at which this takes place is noted. In principle this should correspond to the critical-suction value of the soil sample for drying conditions.

The silica flour was subjected to the above procedure but no clear-cut critical suction value was found. In the flow experiments the critical-suction for wetting conditions is really the one that is applicable. If the flow tube had been set up with tensiometers, this critical-suction value could have been determined. For purposes of calculation a value of 64 cm was used as the critical-suction value (Sec. 7.10).

7.9 Experiments -- Horizontal Flow with Positive Applied Pressure

Figures 7.15 and 7.16 show the moisture profiles for the silica flour with applied pressures of 148 cm and 335 cm, respectively. The lower pressure was achieved by raising the water-supply tank using the fork-lift, while the higher pressure was achieved by admitting air under pressure to the upper part of the tank. In Figs. 7.15 and 7.16, the circles represent the experimental data. Smooth curves were fitted through these data.

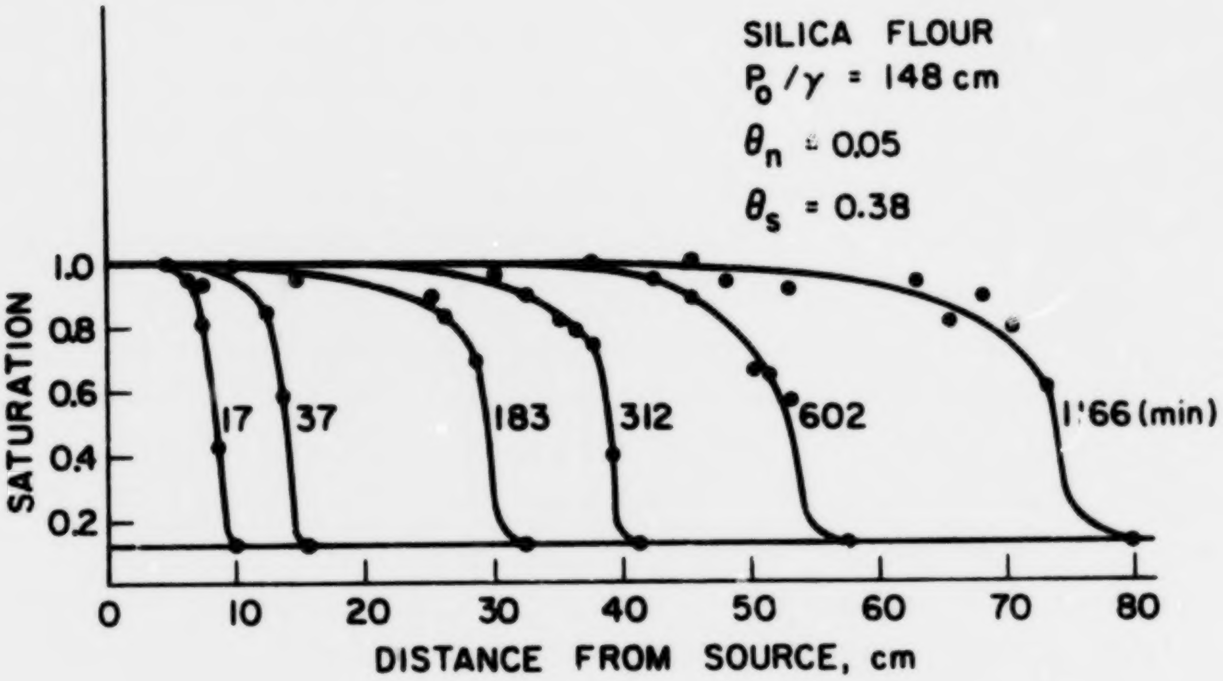

Figure 7.15 Moisture Profile for Silica Flour (p_o/γ = 148 cm).

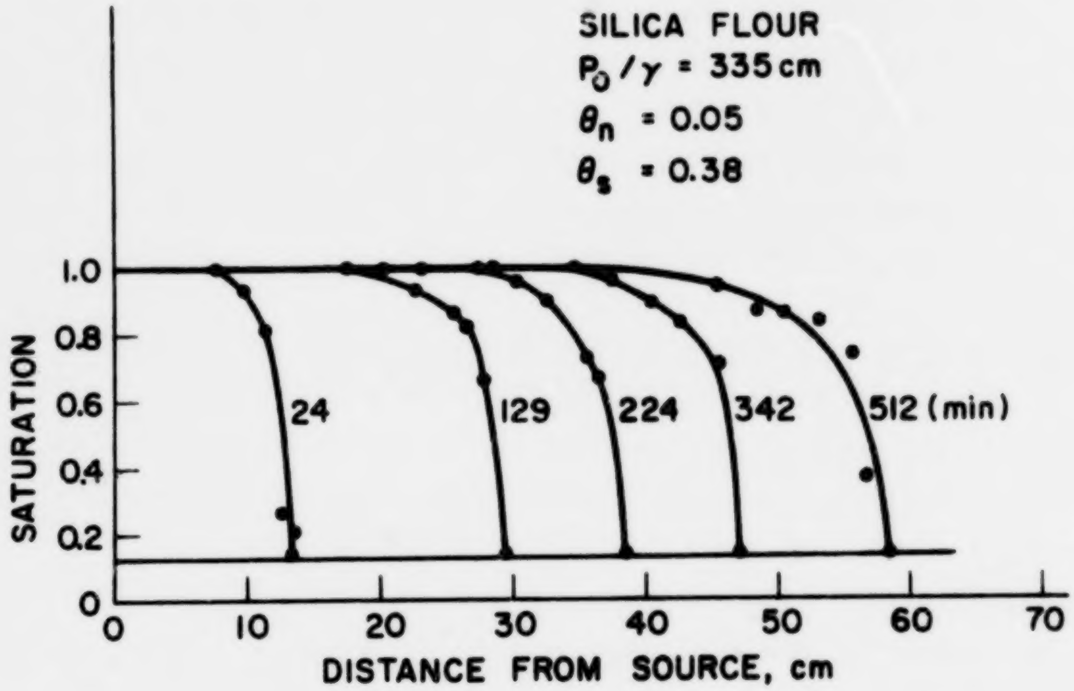

Figure 7.16 Moisture Profile for Silica Flour (p_o/γ = 335 cm).

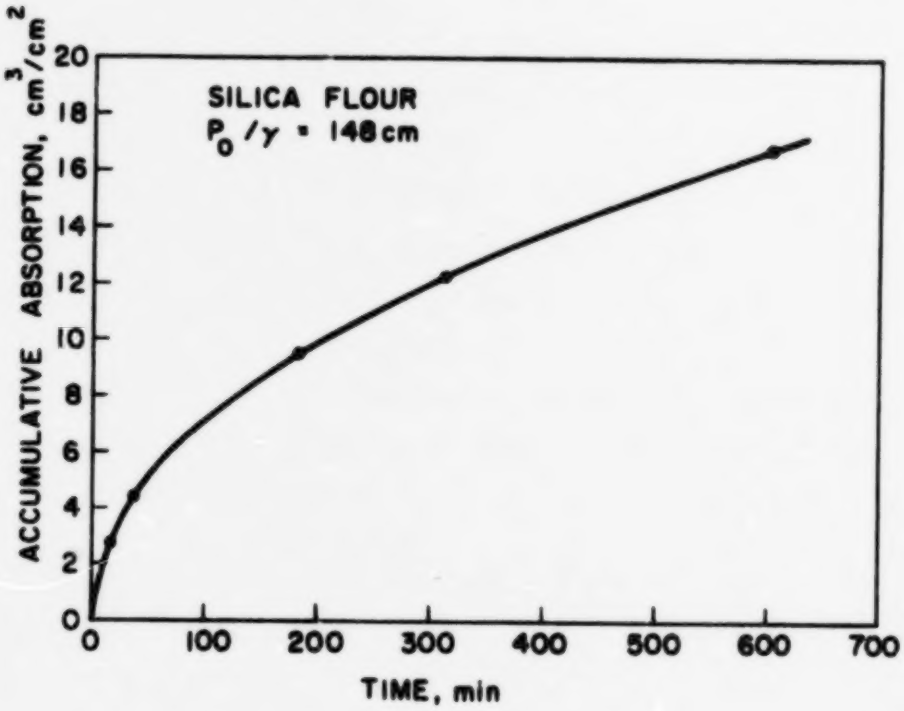

Figure 7.17 Accumulative Absorption for Silica Flour (p_o/γ = 148 cm).

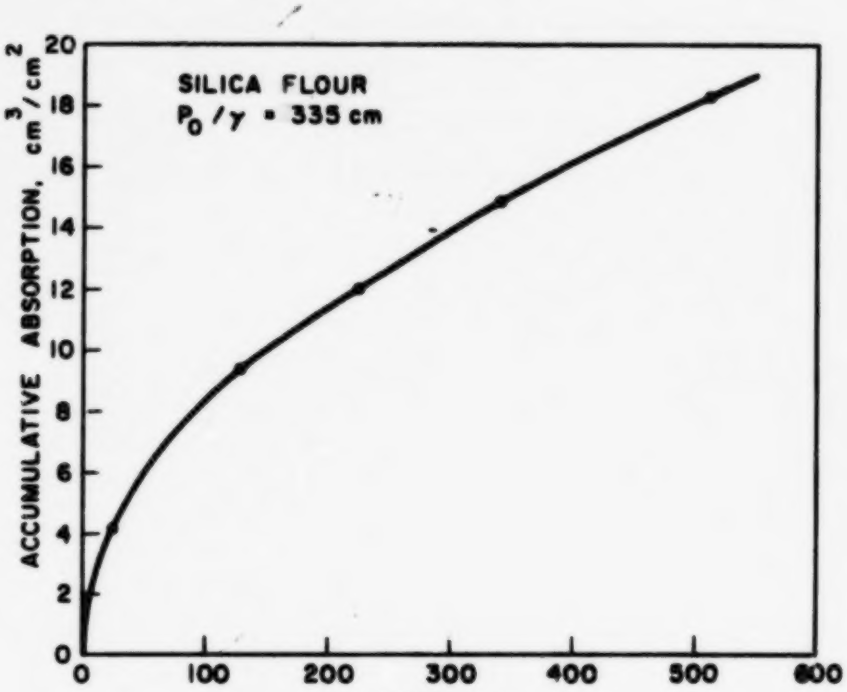

Figure 7.18 Accumulative Absorption for Silica Flour (p_o/γ = 335 cm).

7.10 Calculation of "Effective" Hydraulic Conductivity and Critical-suction Values

Areas under the curves of Figs. 7.15 and 7.16 were planimetered and then converted to accumulative absorption, plotted in Figs. 7.17 and 7.18. The slope at a particular value of time gives the flow velocity V. Figures 7.19 and 7.20 show the flow velocities. By applying Darcy's equation to the flow in the saturated zone, the saturated hydraulic conductivities were found to be 1.9×10^{-3} cm/min and 1.6×10^{-3} cm/min, respectively. These are "effective" values of K_s on the basis of zero critical-suction value. The relation between the true saturated hydraulic conductivity and the "effective" saturated hydraulic conductivity is given by the following expression:

$$V = (K_s)_{true} \frac{P_o/\gamma + S_c}{x_o} = (K_s)_{effective} \frac{P_o/\gamma + 0}{x_o} \qquad (7.8)$$

Using the above relation and the previously mentioned value of 1.33×10^{-3} cm/min for $(K_s)_{true}$ and 1.9×10^{-3} cm/min and 1.6×10^{-3} cm/min for $(K_s)_{effective}$, the corresponding calculated values of the critical suction are 62 cm and 67 cm, respectively. It should be noted that the foregoing procedure could not be applied to the case of zero applied pressure, for in that instance the tension-saturated zone was so small that x_o could not be determined accurately.

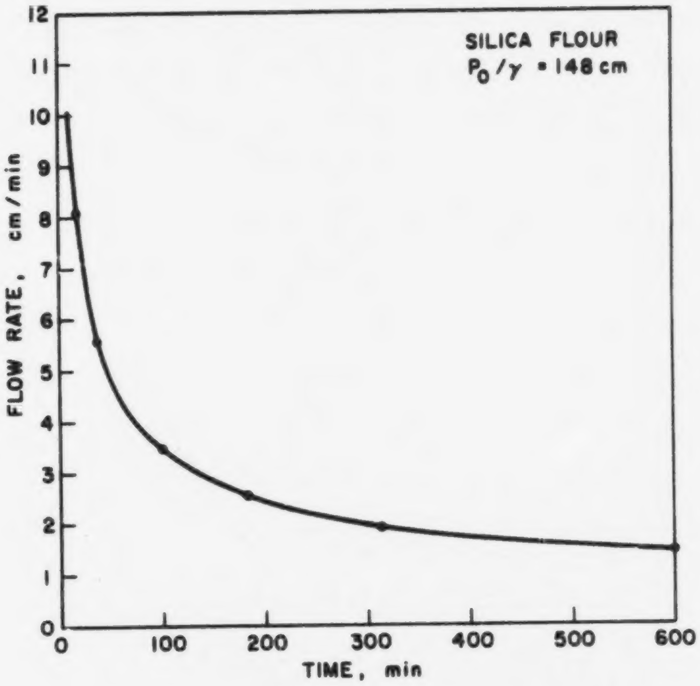

Figure 7.19 Flow Velocity for Silica Flour (p_o/γ = 148 cm).

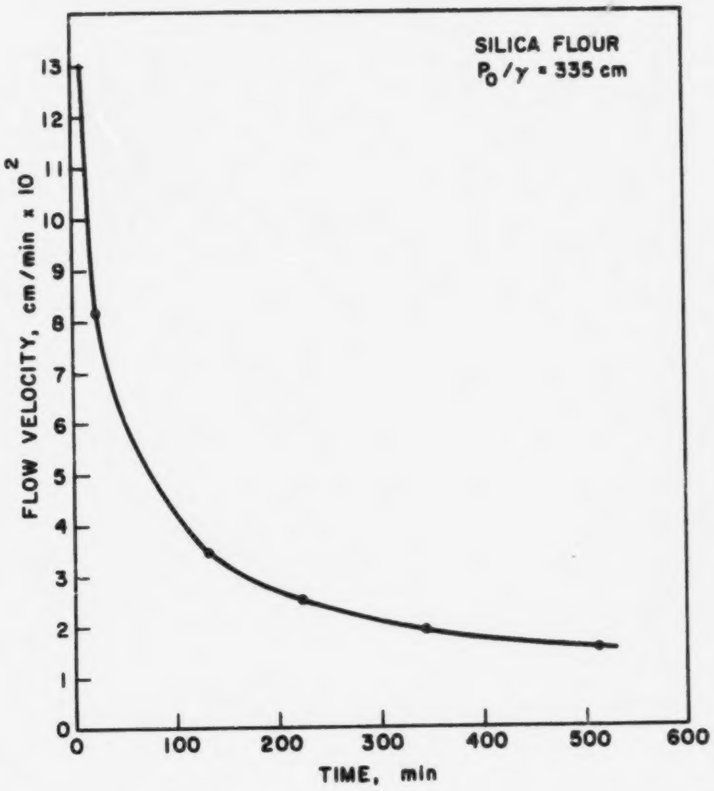

Figure 7.20 Flow Velocity for Silica Flour (p_o/γ = 335 cm).

7.11 Comparison of Experimental Results with Analytical Results

As a check of the applicability of diffusion theory to the description of unsteady unsaturated flow into soils under the influence of applied pressure and the validity of the exponential function of diffusivity, $D = 7.7 \times 10^{-4} e^{25\theta}$ cm^2/min , obtained from the measured data, the moisture profiles were computed by the computer program developed in Chapter 6 using the experimentally determined value for K_s of 1.33×10^{-3} cm/min and a value of 64 cm for S_c as explained in Section 7.10. Figures 7.21, 7.22, and 7.23 show the moisture profiles for three different applied pressures. The solid lines are the computed profiles while the circles represent the experimental data. It should be noted that the computed profiles compare favorably with the experimental points.

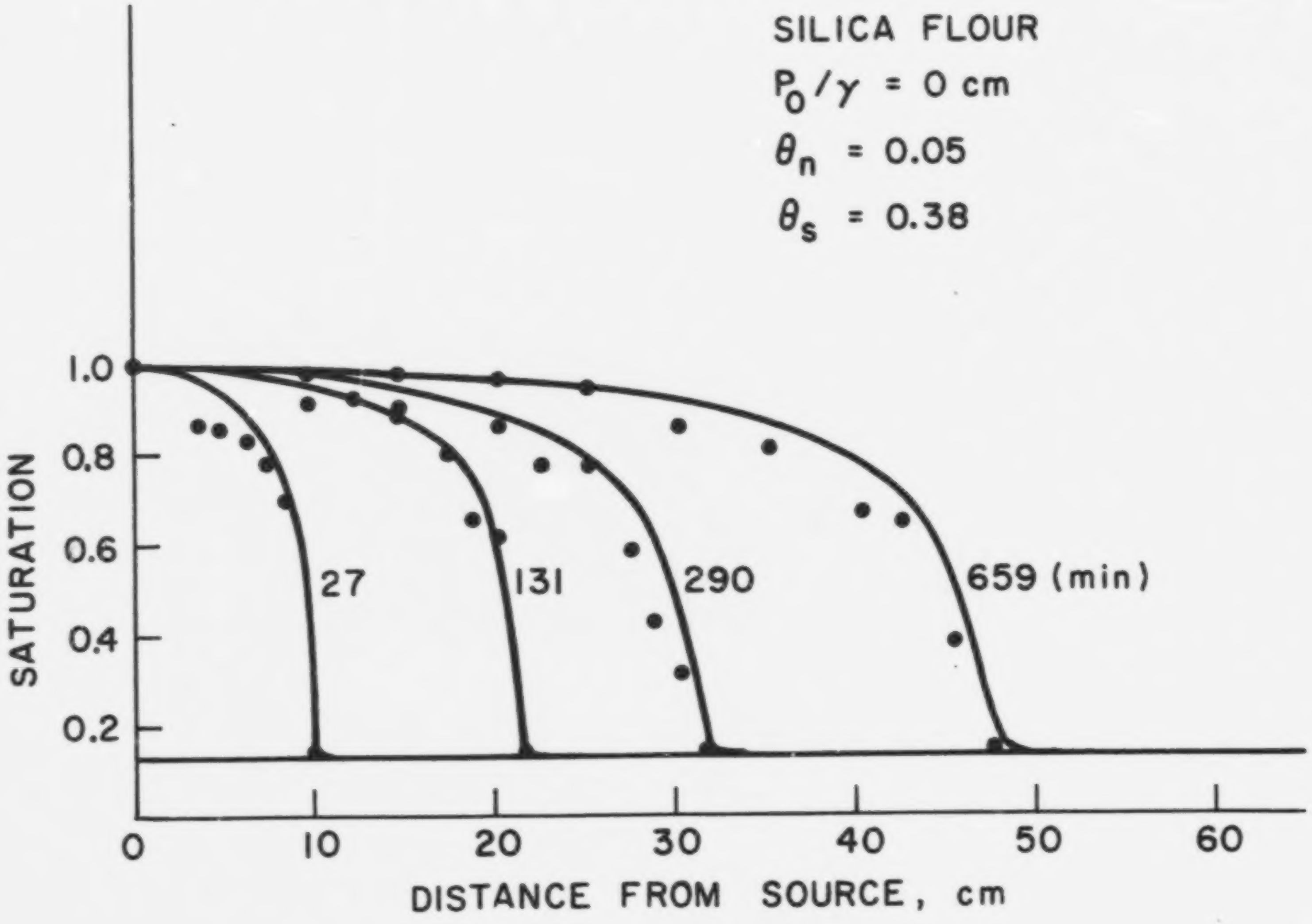

Figure 7.21 Computed Moisture Profiles for Silica Flour (p_o/γ = 0 cm).

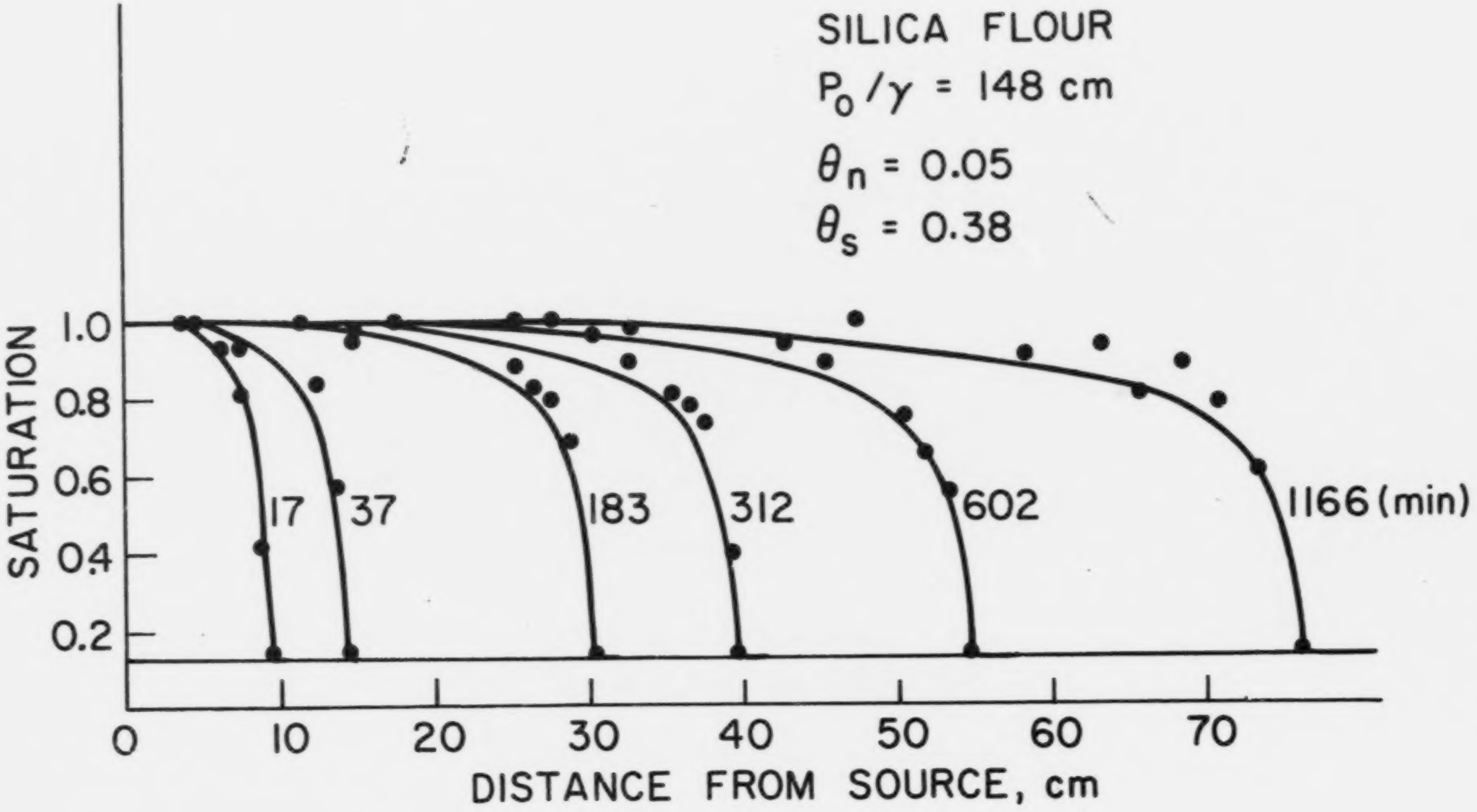

Figure 7.22 Computed Moisture Profiles for Silica Flour (p_o/γ = 148 cm).

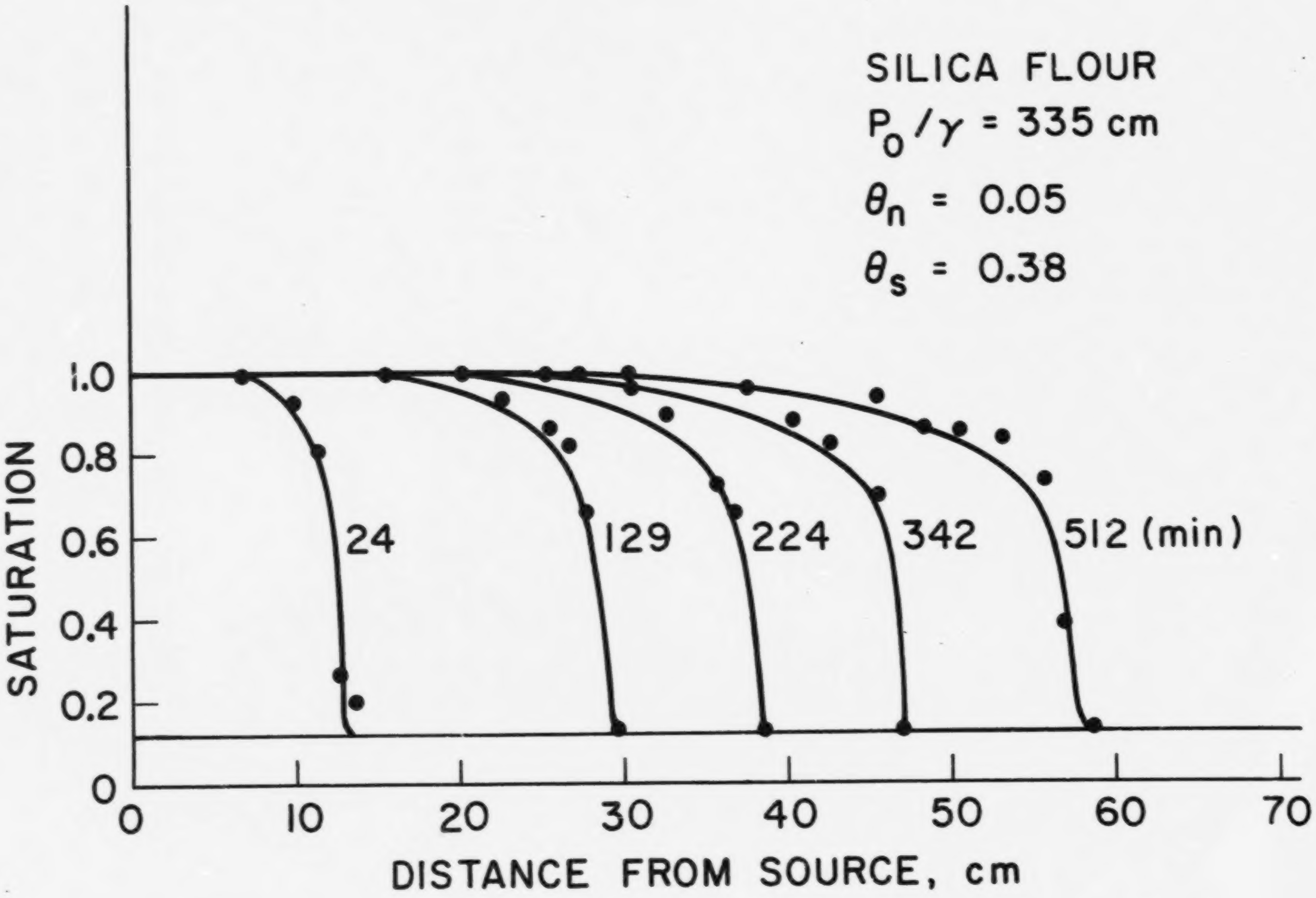

Figure 7.23 Computed Moisture Profiles for Silica Flour (p_o/γ = 335 cm).

CHAPTER 8

DISCUSSION AND CONCLUSION

8.1 Comments on Uniformity of Packings and Comparisons Between the Packings Used in the Tests

It is generally accepted that uniform packing is the key to precise soil-water diffusivity measurements; a non-uniform packing can result in discrepancies in data. The γ-ray attenuation equipment was utilized as a means of checking the uniformity of the packings used in these tests. When the tube is filled with soil, the γ-ray attenuation equation is

$$I_2 = I_o e^{-\left(x_p \tau_p + x_s \tau_s\right)} , \qquad (8.1)$$

where

I_2 = intensity of the transmitted radiation through the tube and soil,

I_o = radiation intensity with no interference,

x_p, x_s = total thickness of plastic and soil traversed by the transmitted γ-ray beams (cm), and

τ_p, τ_s = linear attenuation coefficients for the plastic and soil, respectively.

When the tube is empty, Eq. (8.1) becomes

$$I_1 = I_o e^{-(x_p \tau_p)} \qquad (8.2)$$

where I_1 is the intensity of the transmitted radiation through the empty tube.

Let it be assumed that the statistical fluctuations of the energy of the γ-ray source are independent of the statistical variations of x_p and x_s . It is also assumed that the statistical variations of the thicknesses of plastic and soil are independent of each other and that each have normal distributions, $N\left(m_p,\sigma_p^2\right)$ and $N\left(m_s,\sigma_s^2\right)$, respectively, where m_p and m_s are the mean thicknesses of plastic and soil and σ_p^2 and σ_s^2 are the variances. Assuming the universe average is equal to the sample average, we obtain

$$\overline{I_1} = \frac{1}{N}\sum_{i=1}^{N} I_{1i} \rightarrow m_1 \tag{8.3}$$

$$\overline{I_2} = \frac{1}{N}\sum_{i=1}^{N} I_{2i} \rightarrow m_2 \tag{8.4}$$

where N is the sample size and m represents the sample average.

The variance of I_2 is

$$\sigma_2^2 = \overline{\left(I_2 - \overline{I_2}\right)^2} = \overline{I_2^2} - \left(\overline{I_2}\right)^2$$

$$= \frac{1}{N}\sum_{i=1}^{N} I_{2i}^2 - \left(\overline{I_2}\right)^2$$

$$= \frac{1}{N}\sum_{i=1}^{N} I_{2i}^2 - m_2^2 \tag{8.5}$$

The estimated variance of I_2 is

$$\sigma_2^2 = \widetilde{\left(I_2 - m_2\right)^2}$$

$$= \widetilde{I_1^2\, e^{-2x_s\tau_s}} - m_2^2 \tag{8.6}$$

By independence, σ_2^2 may be expressed as

$$\sigma_2^2 = \widetilde{I_1^2}\;\widetilde{e^{-2x_s\tau_s}} - \widetilde{I_1}^2\left(\widetilde{e^{-x_s\tau_s}}\right)^2 \tag{8.7}$$

Using the Laplace transformation of normal density functions, one gets

$$\sigma_2^2 = \overline{I_1^2}\, e^{-2\tau_s m_s + 2\tau_s^2\sigma_s^2} - \overline{I_1}^2\, e^{-2\tau_s m_s + \tau_s^2\sigma_s^2}$$

or

$$\sigma_2^2 = \left[\overline{I_1^2} - \left(\overline{I_1}\right)^2 e^{-\tau_s^2\sigma_s^2}\right] e^{-2\tau_s m_s + 2\tau_s^2\sigma_s^2} \quad , \qquad (8.8)$$

where m_s is the mean thickness of soil and σ_s^2 represents its variance.

Combining Eqs. (8.1) and (8.2) yields

$$I_2 = I_1 e^{-x_s\tau_s} \qquad (8.9)$$

Therefore,

$$\overline{I_2} = \overline{I_1}\,\overline{e^{-x_s\tau_s}} \qquad (8.10)$$

Once again the Laplace transformation of normal density functions gives

$$\overline{I_2} = \overline{I_1}\, e^{-m_s\tau_s + \frac{1}{2}\tau_s^2\sigma_s^2} \qquad (8.11)$$

Solving Eqs. (8.8) and (8.11) simultaneously permits the calculation of m_s and its variance. The average porosity, p, of soil may be expressed as

$$p = \left(1 - \frac{m_s}{\ell}\right) \qquad (8.12)$$

where ℓ is the distance between the walls of the plastic tube. The variance of p is

$$\text{Var. } p = \left(\frac{1}{\ell}\right)^2 \text{Var. } \left(\ell - m_s\right)^2 = \left(\frac{1}{\ell}\right)^2 \text{Var. } (m_s) \qquad (8.13)$$

Table 8.1 shows the experimental data used in the statistical analysis.

TABLE 8.1

EXPERIMENTAL DATA USED IN THE STATISTICAL ANALYSIS

p_o/γ, cm	τ_s, cm^{-1}	$\overline{I_1^2}$, (cpm)^2	$\overline{I_1}^2$, (cpm)^2	$\overline{I_1}$, cpm	$\overline{I_2}$, cpm	σ_s, cpm
0	0.204	14683377090.0	14683049407.1	121173.6	69050.4	757.7
148	0.204	14683377090.0	14683049407.1	121173.6	69795.7	603.7
335	0.204	14683377090.0	14683049407.1	121173.6	70685.6	656.3

Values of m_s and σ_2 were computed by solving Eqs (8.8) and (8.11) simultaneously. The porosity p and its standard deviation can then be computed from Eqs. (8.12) and (8.13). Table 8.2 shows the results.

TABLE 8.2

COMPARISONS BETWEEN THE PACKINGS USED IN THE EXPERIMENTS

Applied Pressure cm	Porosity, p	Standard Deviation of p
0	0.380	0.011
148	0.392	0.008
335	0.406	0.009

It should be noted that these results compare reasonably well with the experimentally determined overall porosities. The experimental determination of the overall porosity consists of measuring the weight, volume, and specific gravity of the soil sample. The formula used is

$$(1-p)\ V\ SG = W$$

or

$$p = 1 - \frac{W}{V\ SG}$$

where p is the porosity, V is the total volume of sample in cm^3, W is the total weight of the sample in grams, and SG is the specific gravity of the sample in grams/cm^3. The porosities determined by the above procedures were 0.38 for all cases.

8.2 Comparisons Between the Analytical Results and the Simplified Flow Equations

Equations (2.5) and (2.6) are solved for different applied pressures using the capillary suction head, S, as input. Results are plotted in Figs. 8.1, 8.2, amd 8.3. Dashed-lines show the results obtained from the simplified equations. It can be seen from these plots that if one chooses the proper capillary suction head, one can obtain good agreement between the simplified equations and the analytical results. Unfortunately, however, it is extremely difficult to select the proper value of S for a given soil. For example, applying an approximate value for S of 1000 cm (Adrian and Franzini, 1966, Fig. 1), the results are quite different from those given by the analytical results based on the diffusion equation.

8.3 Discussion of Vertical Flow

Figure 8.4 illustrates a simple flow diagram of infiltration into a vertical column under the influence of applied pressure head, p_o/γ. Once again, complete saturation to the wetting front and capillary forces acting concentrated at the wetting front will be assumed.

It is also assumed that the applied pressure remains constant and is distributed uniformly over the surface of the column. After some time t, the saturated wetting front has moved to a distance z. The hydraulic gradient is $-[(p_o/\gamma + z + S)/z]$, so by combining Darcy's equation and the equation of continuity we get

$$\frac{dz}{dt} = \frac{K_s}{n} \frac{z + p_o/\gamma + S}{z}$$

or

$$\left(1 - \frac{p_o/\gamma + S}{z + p_o/\gamma + S}\right) dz = \frac{K_s}{n} dt$$

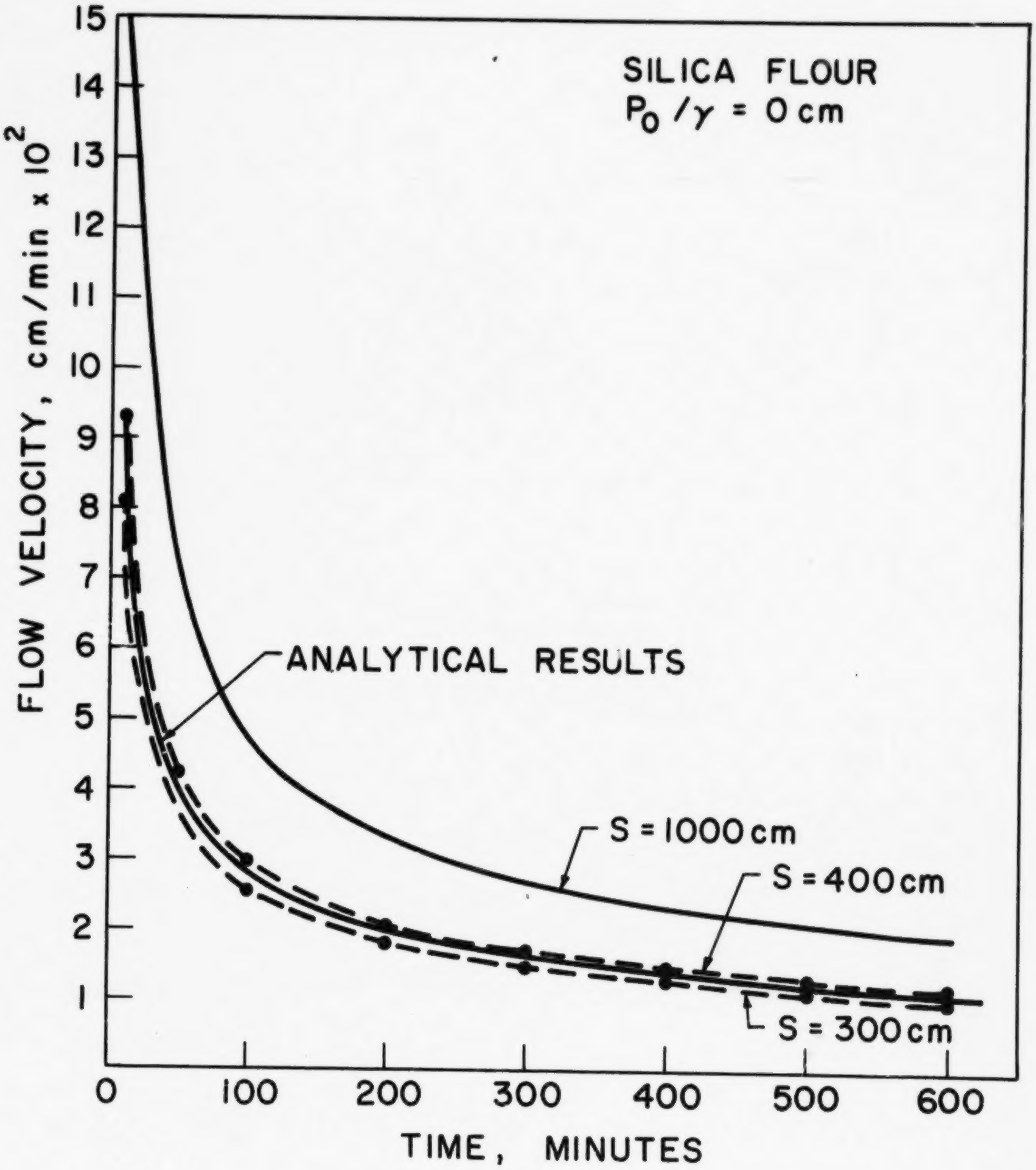

Figure 8.1 Flow Velocity for Silica Flour (p_o/γ = 0 cm).

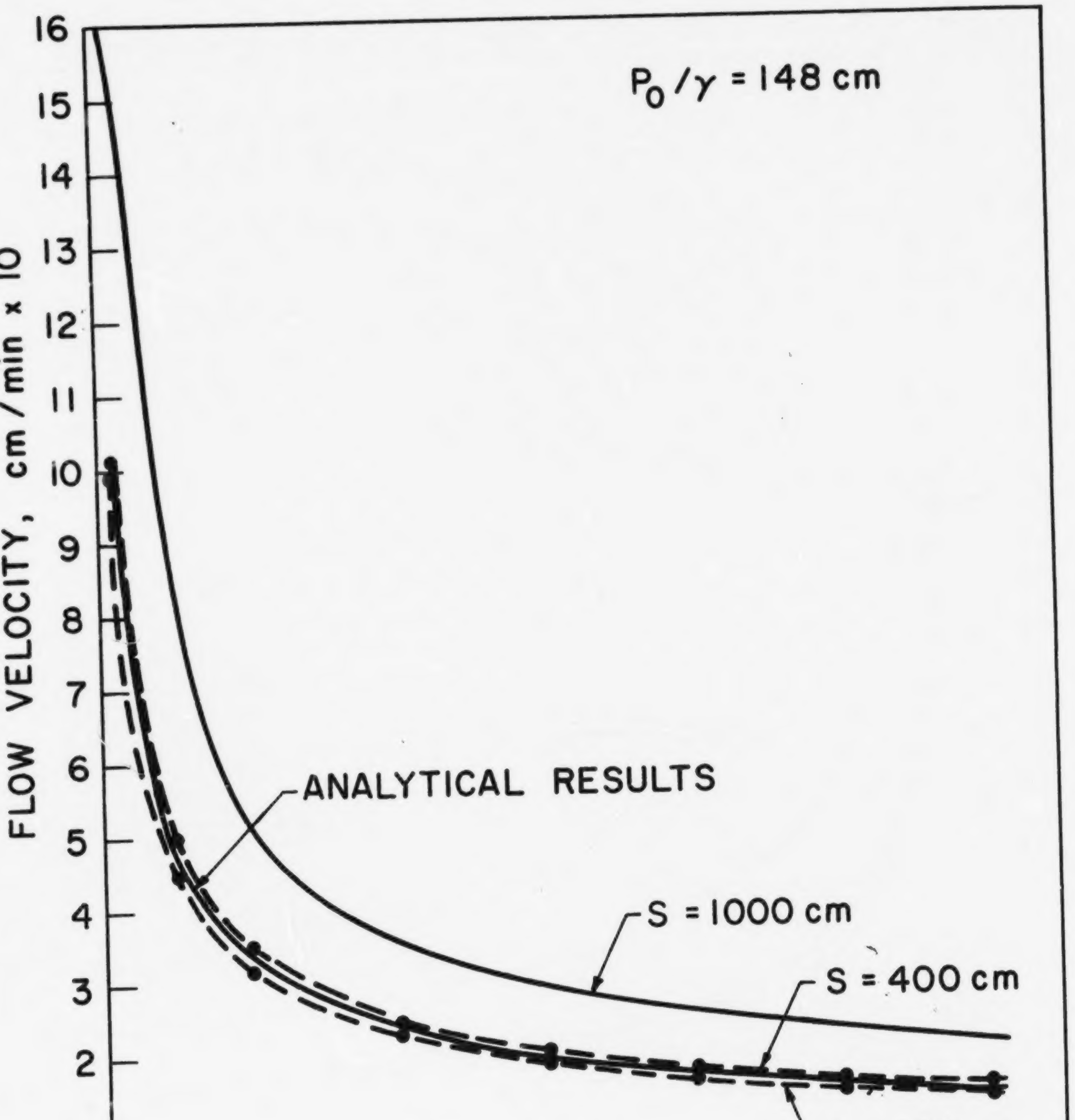

Figure 8.2 Flow Velocity for Silica Flour ($p_o/\gamma = 148$ cm).

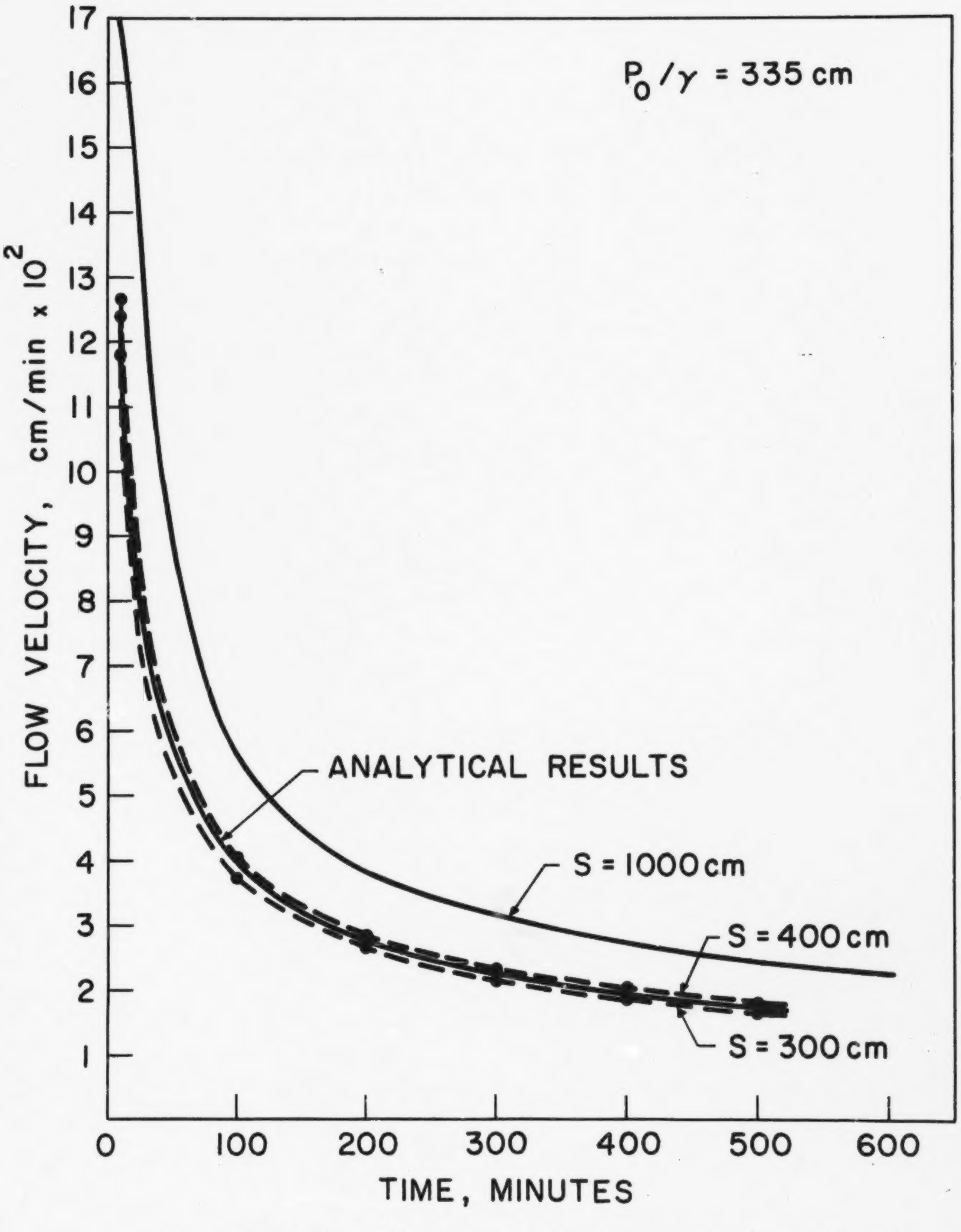

Figure 8.3 Flow Velocity for Silica Flour (p_o/γ = 335 cm).

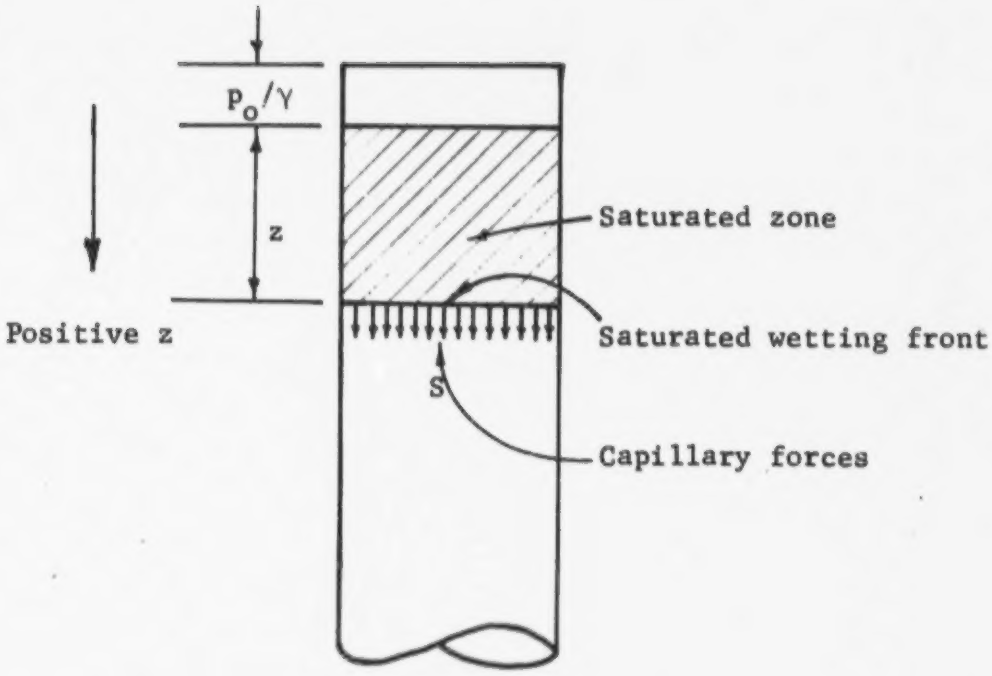
P_o/γ
z
Saturated zone
Positive z
Saturated wetting front
S
Capillary forces

Figure 8.4 Vertical Flow into a Column of Soil Under an Applied Pressure.

The boundary conditions may be expressed as

$$z = 0 \quad \text{when} \quad t = 0$$

$$z = z \quad \text{when} \quad t = t$$

Integrating between these limits yields

$$z + \left(\frac{p_o}{\gamma}+S\right) \ell n \left(\frac{p_o/\gamma + S}{z + p_o/\gamma + S}\right) = \frac{K_s t}{n}$$

Rearranging the terms

$$t = \frac{n}{K_s}\left[z + \left(p_o/\gamma + S\right) \ell n\left(\frac{p_o/\gamma + S}{z + p_o/\gamma + S}\right)\right] \qquad (8.14)$$

which permits the calculation of the time required for the wetting front to move any given value of z. The infiltration rate can be expressed as

$$V = K_s \frac{z + p_o/\gamma + S}{z}$$

which, when rearranged, yields

$$z = \left(p_o/\gamma + S\right) \frac{K_s}{V - K_s} \qquad (8.15)$$

Substituting Eq. (8.15) into Eq. (8.14) gives

$$\frac{tK_s}{n} \frac{1}{\left(p_o/\gamma + S\right)} = \frac{K_s}{V - K_s} + \ell n\left(\frac{1}{1 + \frac{K_s}{V-K_s}}\right) \qquad (8.16)$$

Equation (8.16) is not readily solved in terms of the flow velocity as a function of time. A computer program was developed using the half-interval method. Appendix 2 shows the program and a typical output.

The above procedure will give good results providing one can select a reasonably accurate value of S. For better results, one should use Eq. (4.1).

8.4 Conclusions

In all experiments the position of the wetting front as a function of time was noted by visual observation. Figure 8.5 shows the rate of advance of the wetting front plotted against the square root of

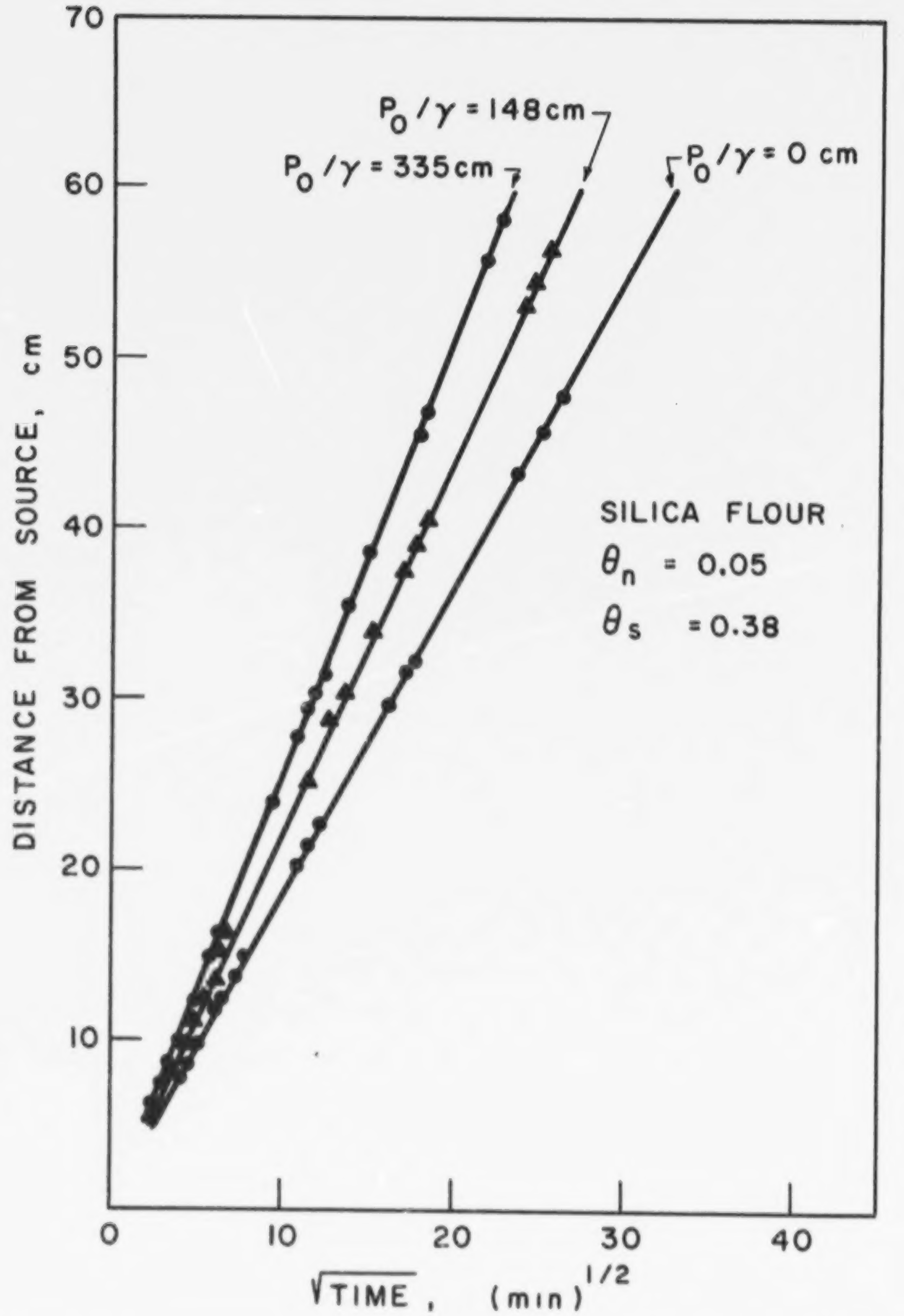

Figure 8.5 Distance to the Wetting Front Versus the Square Root of Time for Silica Flour.

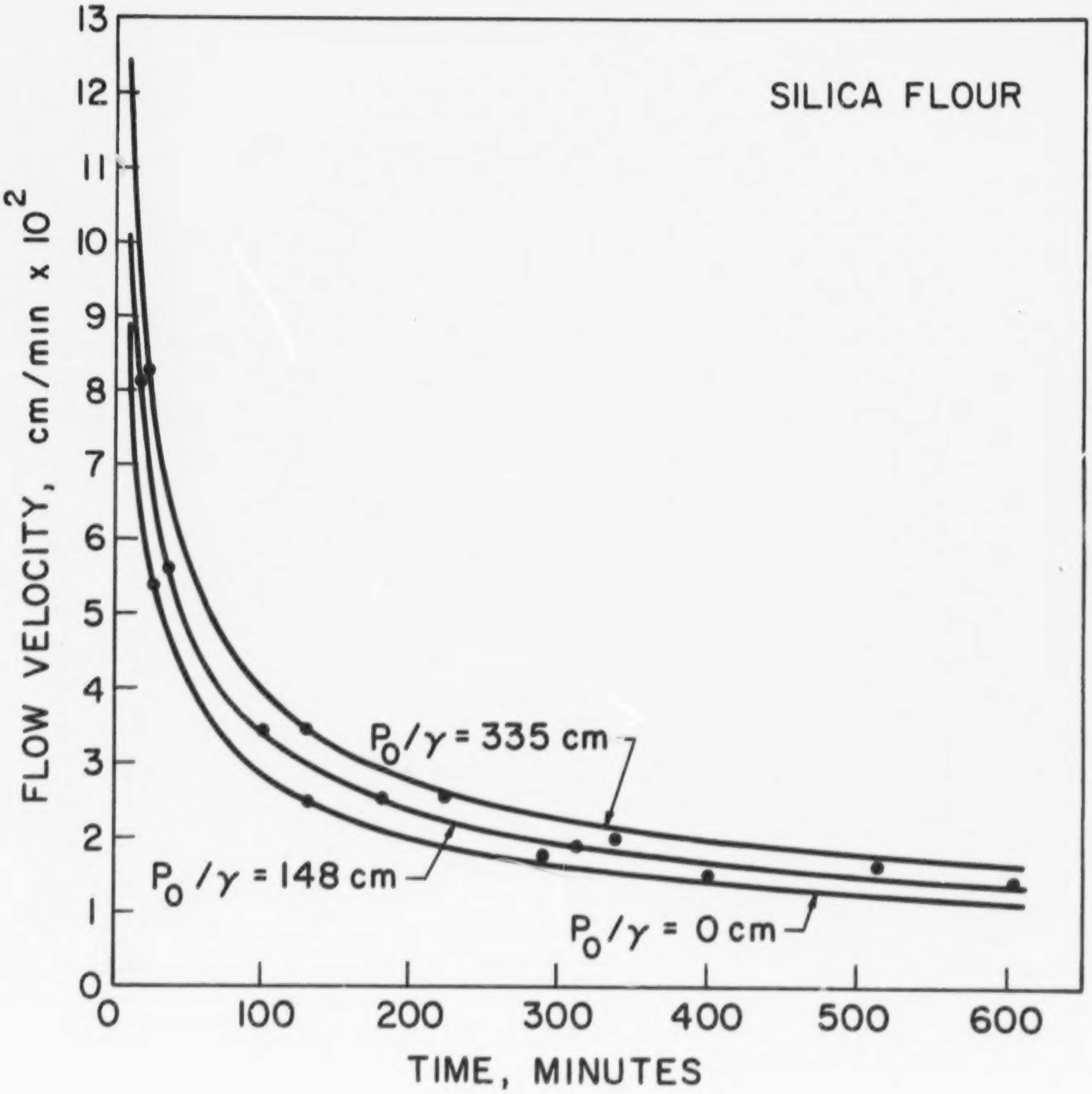

Figure 8.6 Flow Velocity for Silica Flour (versus Time, min).

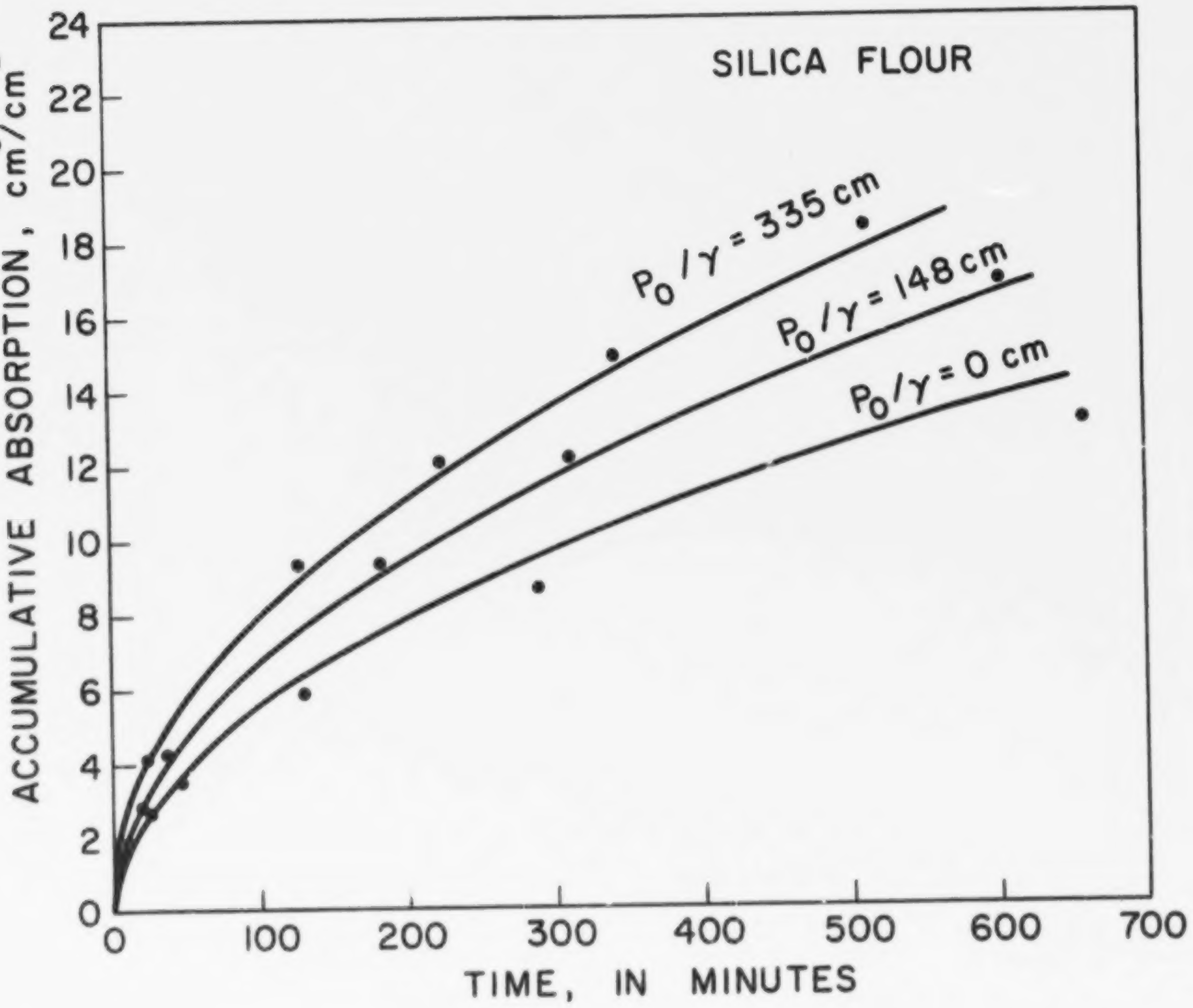

Figure 8.7 Accumulative Absorption for Silica Flour (versus Time, min).

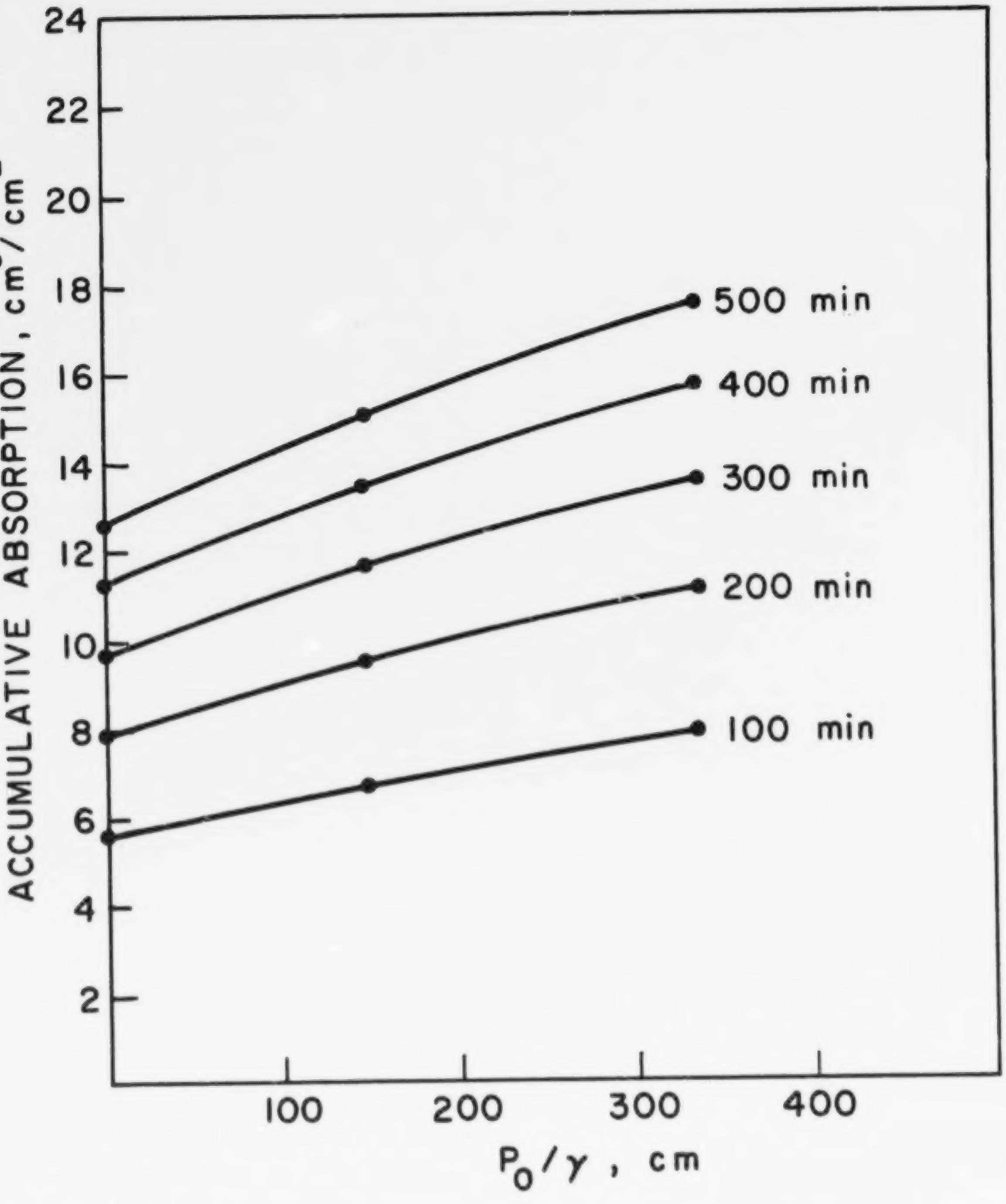

Figure 8.8 Accumulative Absorption for Silica Flour (versus p_o/γ, cm).

time for three different applied pressures. These three lines pass through the origin. The points represent experimental data. It should be noted that the Boltzmann transformation implies a given moisture advance proportional to the square root of time. The visible wetting front is a plane of constant moisture. Therefore, a plot of moisture content versus the square root of time should yield a straight line, as demonstrated in Fig. 8.5. Figure 8.6 shows the flow velocity as a function of time for three different applied pressures. The circles represent experimental data. Figure 8.7 illustrates the accumulative absorption versus time using applied pressure as the parameter, while Fig. 8.8 plots accumulative absorption versus applied pressure with time as the parameter.

In Chapter 6 a theoretical solution has been obtained for moisture movement in a horizontal semi-infinite homogeneous soil column initially at a uniform moisture content under the influence of a certain applied pressure at one end. Application of the theoretical results requires knowledge of the hydraulic characteristics of soil: porosity, saturated hydraulic conductivity, critical-suction value, and the relationship of diffusivity as a function of moisture content. The computer program developed in Chapter 6 applies only for soil having an exponential diffusivity relationship. The procedure can be modified for soil having different diffusivity functions.

Following the experimental determination of the hydraulic characteristics of the soil, a set of experiments was conducted under different applied pressures. Several plots of theoretical and experimental results were presented in Chapters 7 and 8; the agreement between theory and experiment was good.

It has been often pointed out that uniform packing is the key to soil moisture movement experiments. Inconsistencies in experimental results and divergence from theory can be ascribed to non-uniformity in packings. Glass beads were used in early experiments but the results were not very satisfactory because of the slow-growth unsaturated region. Lake Lagunita silty clay, Stanford, was also used; this material was ground and sieved through a #48 sieve. However, it was impossible to get a uniform packing with this soil and the experimental results were very inconsistent. Finally, silica flour was chosen as the soil sample; this was the most satisfactory material.

In Chapter 5, it was assumed that the penetration of the saturated wetting front is proportional to the square root of time. The validity of this assumption was verified in Chapter 7.

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APPENDIX

A.1 Computer Program for Solving Equation (5.21) Subject to (5.37)

```
• • • •              STANFORD B5500 ALGOL -- 07/05/66 VERSION          004/67             6384 • •
BEGIN COMMENT WILLIAM YEH U 621 BOX 173
MOISTURE MOVEMENT IN A HORIZONTAL SOIL COLUMN UNDER THE INFLUENCE
OF AN IMPRESSED PRESSURE, THE FOLLOWING PROGAM SOLVES A SECOND ORDER,
NONLINEAR ORDINARY DIFFERENTIAL EQUATION BY A FOUTH ORDER ADAMS
PREDICTOR CORRECTOR METHOD WITH RUNGE KUTTA FOR STARTING;
FORMAT FMT1(/X10,"PHI=",F20.11);
FORMAT SL(/X10,"DY =",F20.11/),
       HEAD(/X10,"T",X19,"X",X18,"THETA",X14,"SATURATION"),
       HEAD1(4F20.11),
       HEAD2(/4F20.11),
       HEADING(/X9,"X",X20,"Y"/),
       VALUES(2E20.11),
       ER(//X6,"MAX ERROR =",E20.11/);
REAL MAX,MIN,TEST,THETA,RETAIN,M0,M1,M2,M3,H,ERRMAX,ANS,AL,BET,TMX,TMN,
     KSAT,PHI,A,B,C,PRE,X0,I0;
INTEGER I,J,II,K,P;
LABEL BACK,AHEAD,INTERPOL,LAST,RETURN,REPEAT;
REAL ARRAY X[0:999],Y,DY[0:999],YKEAP,DYKEAP[0:10],G[0:2];
COMMENT THE FOLLOWING PROCEDURE EVALUTES, AT A SINGLE ABSCISSA, AN
N TH DEGREE LAGRANGE POLYNOMIAL PASSING THROUGH N+1 POINTS;
PROCEDURE LAGRANGE(N,U,X,Y,ANS);
ARRAY X,Y[1];
INTEGER N; REAL U,ANS;
BEGIN INTEGER I,J; REAL L;
ANS←0;
FOR J←1 STEP 1 UNTIL N+1 DO BEGIN L←1;
FOR I←1 STEP 1 UNTIL N+1 DO
BEGIN IF I≠J THEN L←L×(U-X[I])/(X[J]-X[I]);END;
ANS←ANS+L×Y[J];END;END LAGRANGE;
REAL PROCEDURE M(X,Y,DY);
VALUE X,Y,DY;
REAL X,Y,DY;
M←-DY×(X/(2×Y));
COMMENT ASSIGN VALUES TO CONSTANTS;
    KSAT←0.00133;
    PRE←335+64;
    AL←7.7@-4;
    BET←25;
    TMX←0.380;
    TMN←0.05;
    DY[0]←-348;
    G[1]←-352;
    G[2]←-356;
    II←990;
    H←0.050;
A←TMX-TMN;
J←0;
RETURN:WRITE([PAGE]);
B←SQRT(AL)×DY[0];
C←-PRE×KSAT×BET;
PHI←C/B;
X[0]←PHI/SQRT(AL);
MAX←EXP(BET×TMX);
MIN←EXP(BET×TMN);
```

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```
Y[0]←MAX;                                                                 .   WYEH0055
FOR I←1 STEP 1 UNTIL II DO X[I]←X[0]+I×H;                                 .   WYEH0056
ERRMAX←I←K←0;                                                             .   WYEH0057
       WRITE(SL,DY[I]);                                                   .   WYEH0058
       WRITE(FMT1,PHI);                                                   .   WYEH0059
       WRITE(HEADING);                                                    .   WYEH0060
BACK:  IF I/5 = ENTIER(I/5) THEN                                          .   WYEH0061
       WRITE(VALUES,X[I],Y[I]);                                           .   WYEH0062
       IF I=3 THEN BEGIN I←4;GO TO AHEAD;END;                             .   WYEH0063
COMMENT THE RUNGE KUTTA FORMULAS WILL BE USED TO GET THE FOUR             .   WYEH0064
STARTING VALUES, THEN THE ADAMS PREDICTOR CORRECTOR METHOD IS USED        .   WYEH0065
TO PERFORM THE STEP BY STEP INTEGRATION;                                  .   WYEH0066
    M0←M(X[I],Y[I],DY[I]);                                                .   WYEH0067
    M1←M(X[I]+0.5×H,Y[I]+0.5×H×DY[I],DY[I]+0.5×M0);                       .   WYEH0068
    M2←M(X[I]+0.5×H,Y[I]+0.5×H×DY[I]+0.25×H×M0,DY[I]+0.5×M1);             .   WYEH0069
    M3←M(X[I]+H,Y[I]+H×DY[I]+0.5×H×M1,DY[I]+M2);                          .   WYEH0070
    Y[I+1]←Y[I]+H×DY[I]+H×H×(M0+M1+M2)/6;                                 .   WYEH0071
    DY[I+1]←DY[I]+H×(M0+2×M1+2×M2+M3)/6;                                  .   WYEH0072
    I←I+1;                                                                .   WYEH0073
    GO TO BACK;                                                           .   WYEH0074
AHEAD:DY[I]←DY[I-1]+H×(55×M(X[I-1],Y[I-1],DY[I-1])-59×M(X[I-2],Y[I-2],DY.   WYEH0075
[I-2])+37×M(X[I-3],Y[I-3],DY[I-3])-9×M(X[I-4],Y[I-4],DY[I-4]))/24;        .   WYEH0076
Y[I]←Y[I-1]+H×(55×DY[I-1]-59×DY[I-2]+37×DY[I-3]-9×DY[I-4])/24;            .   WYEH0077
REPEAT:DY[I]←DY[I-1]+H×(9×M(X[I],Y[I],DY[I])+19×M(X[I-1],Y[I-1],DY[I-1]).   WYEH0078
-5×M(X[I-2],Y[I-2],DY[I-2])+M(X[I-3],Y[I-3],DY[I-3]))/24;                 .   WYEH0079
TEST←Y[I];                                                                .   WYEH0080
Y[I]←Y[I-1]+H×(9×DY[I]+19×DY[I-1]-5×DY[I-2]+DY[I-3])/24;                  .   WYEH0081
K←K+1;                                                                    .   WYEH0082
IF K≤1 THEN GO TO REPEAT; K←0;                                            .   WYEH0083
RETAIN←ABS(TEST-Y[I])/18;                                                 .   WYEH0084
IF RETAIN>ERRMAX THEN ERRMAX←RETAIN;                                      .   WYEH0085
IF I/5 = ENTIER(I/5) THEN                                                 .   WYEH0086
WRITE(VALUES,X[I],Y[I]);                                                  .   WYEH0087
I←I+1;                                                                    .   WYEH0088
IF I≤II THEN GO TO AHEAD; I←I-1;                                          .   WYEH0089
IF ABS((Y[I]-Y[I-1])/Y[I-1])>0.0001 THEN BEGIN                            .   WYEH0090
FOR P←1 STEP 1 UNTIL 40 DO X[II+P]←X[II]+P×H;                             .   WYEH0091
II←II+40;I←I+1;GO TO AHEAD;END;                                           .   WYEH0092
YKEAP[J]←Y[I];                                                            .   WYEH0093
DYKEAP[J]←DY[0];                                                          .   WYEH0094
WRITE(ER,ERRMAX);                                                         .   WYEH0095
J←J+1; ERRMAX←0;                                                          .   WYEH0096
IF ABS(Y[I]-MIN)<0.1 THEN GO TO LAST                                      .   WYEH0097
ELSE IF J≥3 THEN GO TO INTERPOL ELSE                                      .   WYEH0098
BEGIN I←0;DY[I]←G[J];GO TO RETURN;END;                                    .   WYEH0099
INTERPOL:LAGRANGE(J,MIN,YKEAP,DYKEAP,ANS);                                .   WYEH0100
I←0;DY[I]←ANS;GO TO RETURN;                                               .   WYEH0101
LAST:BEGIN COMMENT CALCULATION OF MOISTURE PROFILES;                      .   WYEH0102
    FORMAT T(/X40,"T =",F20.11/),                                         .   WYEH0103
           INF(/"AT TIME=",F18.11,X2,"IVOL=",F18.11,X2,"IRATE=",F18.11,.   WYEH0104
           X2,"XSAT=",F18.11/);                                           .   WYEH0105
    INTEGER K,L,M;                                                        .   WYEH0106
    REAL TOTAL,SATU;                                                      .   WYEH0107
    REAL ARRAY MOIST,D,KP[0:4,0:11],R[0:700,0:11],F[0:4];                 .   WYEH0108
    LABEL HARD;                                                           .   WYEH0109
```

```
WRITE([PAGE]);
WRITE(HEAD);
K←L←M←0; I←J←0;
COMMENT VALUES OF TIME,DISTANCE AND THETA ;
MOIST[I,0]←TMX;
FOR I←99 STEP 99 UNTIL II DO
BEGIN
     L←L+1;
     TEST←(LN(Y[I]))/BET;
     MOIST[J,L]←TEST;
END;
L←0;
P←10;
TOTAL←0;
FOR I←0 STEP 1 UNTIL 700 DO R[I,0]←PHI×SQRT(I);
FOR I←1 STEP 1 UNTIL 5,10,24,100,129,200,224,300,342,400,500,512
STEP 1 UNTIL 514 DO
    BEGIN K←K+1;
    M←0;
    WRITE(HEAD2,I, 0,TMX,1);
    FOR J←99 STEP 99 UNTIL II  DO BEGIN
        M←M+1;
        R[K,M]←(X[J]×SQRT(   I×AL));
        SATU←MOIST[0,M]/TMX;
        WRITE(HEAD1,I,R[K,M],MOIST[0,M],SATU);END;
COMMENT CALCULATION OF INFILTRATION;
R[K,0]←PHI×SQRT(I);
X0←(SQRT(I))×PHI;
I0←-(SQRT(AL)/(SQRT(I)×BET))×DY[0];
FOR J←1 STEP 1 UNTIL P   DO
BEGIN
    TOTAL←(R[K,J]-R[K,J-1])×((MOIST[0,J-1]+MOIST[0,J])/2-TMN)
          +TOTAL;
END;
TOTAL←TOTAL+R[I,0]×(TMX-TMN);
WRITE(INF,I,TOTAL,I0,X0);
TOTAL←0;
    END;
END;END.
```

```
          T                      X                     THETA                SATURATION

     1.00000000000         0.00000000000         0.38000000000         1.00000000000
     1.00000000000         1.48975816689         0.37441357365         0.98529887804
     1.00000000000         1.62711511244         0.36799886425         0.96841806382
     1.00000000000         1.76447203800         0.36047849450         0.94862761710
     1.00000000000         1.90182896356         0.35140859948         0.92475947232
     1.00000000000         2.03918586912         0.34001250097         0.89476973939
     1.00000000000         2.17654281469         0.32473689252         0.85457076978
     1.00000000000         2.31389974026         0.30168817190         0.79391624184
     1.00000000000         2.45125666584         0.25475492295         0.67040769197
     1.00000000000         2.58861359136         0.04902236952         0.12900623557
     1.00000000000         2.72597051694         0.04902236951         0.12900623555
AT TIME=     1.00000000000  IVOL=     0.78130947052  IRATE=     0.39239093839  XSAT=     1.35240126132

     2.00000000000         0.00000000000         0.38000000000         1.00000000000
     2.00000000000         2.10683623253         0.37441357365         0.98529887804
     2.00000000000         2.30103825956         0.36799886425         0.96841806382
     2.00000000000         2.49534028654         0.36047849450         0.94862761710
     2.00000000000         2.68959231357         0.35140859948         0.92475947232
     2.00000000000         2.88384434060         0.34001250097         0.89476973939
     2.00000000000         3.07809636761         0.32473689252         0.85457076978
     2.00000000000         3.27234839463         0.30168817190         0.79391624184
     2.00000000000         3.46660042166         0.25475492295         0.67040769197
     2.00000000000         3.66085244865         0.04902236952         0.12900623557
     2.00000000000         3.85510447568         0.04902236951         0.12900623555
AT TIME=     2.00000000000  IVOL=     1.10493844961  IRATE=     0.27746229341  XSAT=     1.91258420552

     3.00000000000         0.00000000000         0.38000000000         1.00000000000
     3.00000000000         2.58033687067         0.37441357365         0.98529887804
     3.00000000000         2.81824604451         0.36799886425         0.96841806382
     3.00000000000         3.05615521836         0.36047849450         0.94862761710
     3.00000000000         3.29406439219         0.35140859948         0.92475947232
     3.00000000000         3.53197356605         0.34001250097         0.89476973939
     3.00000000000         3.76988273990         0.32473689252         0.85457076978
     3.00000000000         4.00779191375         0.30168817190         0.79391624184
     3.00000000000         4.24570108761         0.25475492295         0.67040769197
     3.00000000000         4.48361026142         0.04902236952         0.12900623557
     3.00000000000         4.72151943528         0.04902236951         0.12900623555
AT TIME=     3.00000000000  IVOL=     1.35326769936  IRATE=     0.22654701390  XSAT=     2.34242769684

     4.00000000000         0.00000000000         0.38000000000         1.00000000000
     4.00000000000         2.97951637376         0.37441357365         0.98529887804
     4.00000000000         3.25423022489         0.36799886425         0.96841806382
     4.00000000000         3.52894407601         0.36047849450         0.94862761710
     4.00000000000         3.80365792711         0.35140859948         0.92475947232
     4.00000000000         4.07837177826         0.34001250097         0.89476973939
     4.00000000000         4.35308562939         0.32473689252         0.85457076978
```

```
      4.00000000000        4.62779949053        0.30168817190        0.79391624184
      4.00000000000        4.90251333166        0.25475492295        0.67040769197
      4.00000000000        5.17722714275        0.04902236952        0.12900623557
      4.00000000000        5.45194103388        0.04902236951        0.12900623555
AT TIME=     4.00000000000  IVOL=     1.56261894104  IRATE=     0.19619546910  XSAT=     2.70480252264

      5.00000000000        0.00000000000        0.38000000000        1.00000000000
      5.00000000000        3.33120057589        0.37441357365        0.98529887804
      5.00000000000        3.63833999864        0.36799886425        0.96841806382
      5.00000000000        3.94547942139        0.36047849450        0.94862761710
      5.00000000000        4.25261884408        0.35140859948        0.92475947232
      5.00000000000        4.55975826684        0.34001250097        0.89476973939
      5.00000000000        4.86689768959        0.32473689252        0.85457076978
      5.00000000000        5.17403711235        0.30168817190        0.79391624184
      5.00000000000        5.48117653509        0.25475492295        0.67040769197
      5.00000000000        5.78831595780        0.04902236952        0.12900623557
      5.00000000000        6.09545538050        0.04902236951        0.12900623555
AT TIME=     5.00000000000  IVOL=     1.74706108753  IRATE=     0.17548256240  XSAT=     3.02406115316

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     10.00000000000        5.14538977059        0.36799886425        0.96841806382
     10.00000000000        5.57975050780        0.36047849450        0.94862761710
     10.00000000000        6.01411124490        0.35140859948        0.92475947232
     10.00000000000        6.44847198210        0.34001250097        0.89476973939
     10.00000000000        6.88283271930        0.32473689252        0.85457076978
     10.00000000000        7.31719345650        0.30168817190        0.79391624184
     10.00000000000        7.75155419370        0.25475492295        0.67040769197
     10.00000000000        8.18591493080        0.04902236952        0.12900623557
     10.00000000000        8.62027566800        0.04902236951        0.12900623555
AT TIME=    10.00000000000  IVOL=     2.47071748429  IRATE=     0.12408490985  XSAT=     4.27666829625

     24.00000000000        0.00000000000        0.38000000000        1.00000000000
     24.00000000000        7.29829479600        0.37441357365        0.98529887804
     24.00000000000        7.97120355650        0.36799886425        0.96841806382
     24.00000000000        8.64411231700        0.36047849450        0.94862761710
     24.00000000000        9.31702107750        0.35140859948        0.92475947232
     24.00000000000        9.98992983810        0.34001250097        0.89476973939
     24.00000000000       10.66283859860        0.32473689252        0.85457076978
     24.00000000000       11.33574735920        0.30168817190        0.79391624184
     24.00000000000       12.00865611980        0.25475492295        0.67040769197
     24.00000000000       12.68156488020        0.04902236952        0.12900623557
     24.00000000000       13.35447364080        0.04902236951        0.12900623555
AT TIME=    24.00000000000  IVOL=     3.62761906795  IRATE=     0.08009646489  XSAT=     6.62538603550

    100.00000000000        0.00000000000        0.38000000000        1.00000000000
    100.00000000000       14.49758166880        0.37441357365        0.98529887804
```

NOT REPRODUCIBLE

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      100.00000000000        16.27115112450        0.36799886425        0.96841806382
      100.00000000000        17.64472038010        0.36047849450        0.94862761710
      100.00000000000        19.01828963550        0.35140859948        0.92475947232
      100.00000000000        20.39185889120        0.34001250097        0.89476973939
      100.00000000000        21.76542814690        0.32473689252        0.85457074978
      100.00000000000        23.13899740260        0.30168817190        0.79391624184
      100.00000000000        24.51256665830        0.25475492295        0.67040769197
      100.00000000000        25.88613591380        0.04902236952        0.12900623557
      100.00000000000        27.25970516950        0.04902236951        0.12900623555
AT TIME=   100.00000000000   IVOL=     7.81309470520   IRATE=     0.03923909384   XSAT=     13.52401261320

      129.00000000000         0.00000000000        0.38000000000        1.00000000000
      129.00000000000        16.92040040130        0.37441357365        0.98529887804
      129.00000000000        18.48047518320        0.36799886425        0.96841806382
      129.00000000000        20.04054996510        0.36047849450        0.94862761710
      129.00000000000        21.60062474690        0.35140859948        0.92475947232
      129.00000000000        23.16069952880        0.34001250097        0.89476973939
      129.00000000000        24.72077431060        0.32473689252        0.85457074978
      129.00000000000        26.28084909260        0.30168817190        0.79391624184
      129.00000000000        27.84092387450        0.25475492295        0.67040769197
      129.00000000000        29.40099865620        0.04902236952        0.12900623557
      129.00000000000        30.96107343810        0.04902236951        0.12900623555
AT TIME=   129.00000000000   IVOL=     8.87396974560   IRATE=     0.03454804573   XSAT=     15.36032561960

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      200.00000000000        21.06836232530        0.37441357365        0.98529887804
      200.00000000000        23.01088259560        0.36799886425        0.96841806382
      200.00000000000        24.95340286570        0.36047849450        0.94862761710
      200.00000000000        26.89592313570        0.35140859948        0.92475947232
      200.00000000000        28.83844340590        0.34001250097        0.89476973939
      200.00000000000        30.78096367620        0.32473689252        0.85457074978
      200.00000000000        32.72348394640        0.30168817190        0.79391624184
      200.00000000000        34.66600421660        0.25475492295        0.67040769197
      200.00000000000        36.60852448650        0.04902236952        0.12900623557
      200.00000000000        38.55104475680        0.04902236951        0.12900623555
AT TIME=   200.00000000000   IVOL=    11.04938449620   IRATE=     0.02774622934   XSAT=     19.12584205530

      224.00000000000         0.00000000000        0.38000000000        1.00000000000
      224.00000000000        22.29655869780        0.37441357365        0.98529887804
      224.00000000000        24.35242911840        0.36799886425        0.96841806382
      224.00000000000        26.40819933890        0.36047849450        0.94862761710
      224.00000000000        28.46396955940        0.35140859948        0.92475947232
      224.00000000000        30.51973978010        0.34001250097        0.89476973939
      224.00000000000        32.57551000080        0.32473689252        0.85457074978
      224.00000000000        34.63128022150        0.30168817190        0.79391624184
      224.00000000000        36.68705044220        0.25475492295        0.67040769197
      224.00000000000        38.74282066260        0.04902236952        0.12900623557
      224.00000000000        40.79859068330        0.04902236951        0.12900623555
```

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AT TIME=   224.00000000000  IVOL=     11.69356940690  IRATE=      0.02621772238  XSAT=    20.24088867730

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   300.00000000000     28.18246044510      0.36799886425      0.96841804382
   300.00000000000     30.56155218360      0.36047849450      0.94862761710
   300.00000000000     32.94064392180      0.35140859948      0.92475947232
   300.00000000000     35.31973566050      0.34001250097      0.89476973939
   300.00000000000     37.69882739890      0.32473689252      0.85457074978
   300.00000000000     40.07791913750      0.30168817190      0.79391624184
   300.00000000000     42.45701087600      0.25475492295      0.67040760197
   300.00000000000     44.83610261420      0.04902236952      0.12900623557
   300.00000000000     47.21519435270      0.04902236951      0.12900623555
AT TIME=   300.00000000000  IVOL=     13.53267699380  IRATE=      0.02265470130  XSAT=    23.42427696830

   342.00000000000      0.00000000000      0.38000000000      1.00000000000
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   342.00000000000     30.09063355070      0.36799886425      0.96841804382
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   342.00000000000     35.17098328250      0.35140859948      0.92475947232
   342.00000000000     37.71115814860      0.34001250097      0.89476973939
   342.00000000000     40.25133301470      0.32473689252      0.85457074978
   342.00000000000     42.79150788080      0.30168817190      0.79391624184
   342.00000000000     45.33168274700      0.25475492295      0.67040760197
   342.00000000000     47.87185761260      0.04902236952      0.12900623557
   342.00000000000     50.41203247870      0.04902236951      0.12900623555
AT TIME=   342.00000000000  IVOL=     14.44894512210  IRATE=      0.02121807189  XSAT=    25.01028381880

   400.00000000000      0.00000000000      0.38000000000      1.00000000000
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   400.00000000000     35.28944076000      0.36047849450      0.94862761710
   400.00000000000     38.03657927110      0.35140859948      0.92475947232
   400.00000000000     40.78371778240      0.34001250097      0.89476973939
   400.00000000000     43.53085629370      0.32473689252      0.85457074978
   400.00000000000     46.27799480510      0.30168817190      0.79391624184
   400.00000000000     49.02513331650      0.25475492295      0.67040760197
   400.00000000000     51.77227182730      0.04902236952      0.12900623557
   400.00000000000     54.51941033870      0.04902236951      0.12900623555
AT TIME=   400.00000000000  IVOL=     15.62618941030  IRATE=      0.01961954692  XSAT=    27.04802522650

   500.00000000000      0.00000000000      0.38000000000      1.00000000000
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   500.00000000000     36.38339998630      0.36799886425      0.96841804382
   500.00000000000     39.45479421360      0.36047849450      0.94862761710
   500.00000000000     42.52618844090      0.35140859948      0.92475947232
   500.00000000000     45.59758266830      0.34001250097      0.89476973939
   500.00000000000     48.66897689590      0.32473689252      0.85457074978
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NOT REPRODUCIBLE

```
     500.00000000000       51.74037112330       0.30168817190       0.79391624184
     500.00000000000       54.81176535090       0.25475492295       0.67040769197
     500.00000000000       57.88315957800       0.04902236952       0.12900623557
     500.00000000000       60.95455380500       0.04902236951       0.12900623555

 AT TIME=   500.00000000000  IVOL=    17.47061087540  IRATE=     0.01754825624  XSAT=    30.24061153170

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     512.00000000000       39.92544458500       0.36047849450       0.94862761710
     512.00000000000       43.03347701710       0.35140859948       0.92475947232
     512.00000000000       46.14150944950       0.34001250097       0.89476973939
     512.00000000000       49.24954188180       0.32473689252       0.85457076978
     512.00000000000       52.35757431420       0.30168817190       0.79391624184
     512.00000000000       55.46560674700       0.25475492295       0.67040769197
     512.00000000000       58.57363917800       0.04902236952       0.12900623557
     512.00000000000       61.68167161100       0.04902236951       0.12900623555
 AT TIME=   512.00000000000  IVOL=    17.67901519380  IRATE=     0.01734139234  XSAT=    30.60134728860

     513.00000000000        0.00000000000       0.38000000000       1.00000000000
     513.00000000000       33.74228297850       0.37441357365       0.98529887804
     513.00000000000       36.85334911810       0.36799886425       0.96841804382
     513.00000000000       39.96441525760       0.36047849450       0.94862761710
     513.00000000000       43.07548139710       0.35140859948       0.92475947232
     513.00000000000       46.18654753690       0.34001250097       0.89476973939
     513.00000000000       49.29761367660       0.32473689252       0.85457076978
     513.00000000000       52.40867981630       0.30168817190       0.79391624184
     513.00000000000       55.51974595600       0.25475492295       0.67040769197
     513.00000000000       58.63081209500       0.04902236952       0.12900623557
     513.00000000000       61.74187823500       0.04902236951       0.12900623555
 AT TIME=   513.00000000000  IVOL=    17.69627143540  IRATE=     0.01732448315  XSAT=    30.63121683920

     514.00000000000        0.00000000000       0.38000000000       1.00000000000
     514.00000000000       33.77515418250       0.37441357365       0.98529887804
     514.00000000000       36.88925107410       0.36799886425       0.96841804382
     514.00000000000       40.00334796570       0.36047849450       0.94862761710
     514.00000000000       43.11744485690       0.35140859948       0.92475947232
     514.00000000000       46.23154174860       0.34001250097       0.89476973939
     514.00000000000       49.34563864020       0.32473689252       0.85457076978
     514.00000000000       52.45973553190       0.30168817190       0.79391624184
     514.00000000000       55.57383242400       0.25475492295       0.67040769197
     514.00000000000       58.68792931500       0.04902236952       0.12900623557
     514.00000000000       61.80202620600       0.04902236951       0.12900623555
 AT TIME=   514.00000000000  IVOL=    17.71351086590  IRATE=     0.01730762234  XSAT=    30.66105729120
```

A.2 Computer Program for Solving Equation (8.16)

```
• • • .                STANFORD B5500 ALGOL -- 07/05/66 VERSION         055/67
BEGIN COMMENT COMPARISIONS BETWEEN THE PACKINGS USED IN THE TESTS;
REAL MS,TAUS,SIGS,I1B,I2B,SIG2,I1SB,I1BS,Y1,Y2,Y3,SIGS1,SIGS2,SIGS3,
     N,VAR,SD;
LABEL L1,L2,L3;
FORMAT FMT1(X10,"MS",X15,"SDMS",X12,"N",X15,"SDN");
L1: READ(SIG2,I2B);
I1SB←14683377090.0;
I1BS←14683049407.1;
I1B←121173.6333;
TAUS←0.204;
SIGS1←0.000;
SIGS2←0.400;
L3: Y1←(SIG2×SIG2)-(I1SB-I1BS×EXP(-TAUS×TAUS×SIGS1×SIGS1))×
        (EXP(TAUS×TAUS×SIGS1×SIGS1-2×LN(I1B/I2B)));
     Y2←(SIG2×SIG2)-(I1SB-I1BS×EXP(-TAUS×TAUS×SIGS2×SIGS2))×
        (EXP(TAUS×TAUS×SIGS2×SIGS2-2×LN(I1B/I2B)));
IF ABS(SIGS1-SIGS2)<0.00001 THEN
     BEGIN SIGS←SIGS2; GO TO L2; END;
SIGS3←0.5×(SIGS1+SIGS2);
     Y3←(SIG2×SIG2)-(I1SB-I1BS×EXP(-TAUS×TAUS×SIGS3×SIGS3))×
        (EXP(TAUS×TAUS×SIGS3×SIGS3-2×LN(I1B/I2B)));
IF Y3=0 THEN
   BEGIN SIGS←SIGS3; GO TO L2; END;
IF (Y1×Y3)<0 THEN
BEGIN SIGS2←SIGS3; GO TO L3;END;
SIGS1←SIGS3; GO TO L3;
L2:MS←0.5×(TAUS×SIGS×SIGS)+(1/TAUS)×LN(I1B/I2B);
N←(4.445-MS)/4.445;
VAR←(1/4.445)*2×(SIGS×SIGS);
SD←SQRT(VAR);
WRITE(FMT1);
WRITE([DBL]);
WRITE(<X5,4(F12.5,X4)>,MS,SIGS,N,SD);
WRITE([PAGE]);
GO TO L1;
END.
```

A.3 Propagation of Errors

The expected standard deviation σ for radioactive disintegration is

$$\sigma = \left(Me^{-\lambda t} \right)^{\frac{1}{2}} \tag{A.1}$$

where M is the expected number of atoms disintegrating in the time t , t represents time, and λ is a constant.

In counting practice λt is very small; that is, the observation time t is short compared to the half-life of the radioactive material, therefore we can write

$$\sigma = (M)^{\frac{1}{2}}$$

If a reasonably large number m of counts has been obtained, that number m may be used in the place of M for the purpose of evaluating σ . Then, the standard deviation is

$$\sigma = (m)^{\frac{1}{2}} = (\text{recorded number of counts})^{\frac{1}{2}} \tag{A.2}$$

Example 1: If 100 counts are recorded in 1 minute, the standard deviation is

$$\sigma = (100)^{\frac{1}{2}} = 10 \quad ,$$

and the counting rate might be written 100 ± 10 cpm (counts per minute). If 1000 counts are recorded in 10 minutes, the standard deviation is $\sigma = (1000)^{\frac{1}{2}} = 32$; the counting rate is

$$\frac{1000 \pm 32}{10} = 100 \pm 3.2 \text{ cpm} \quad .$$

From the above example it is clear that for a given counting rate R the standard deviation σ for the rate is inversely proportional to the square root of the time of measurement:

$$R = \frac{m}{t}$$

$$\sigma_R = \frac{(m)^{\frac{1}{2}}}{t} = \frac{(Rt)^{\frac{1}{2}}}{t} = (R/t)^{\frac{1}{2}} \qquad \text{(A.3)}$$

Frequently, it is necessary to work with the addition, subtraction, multiplication or quotient of counting rates. The expressions for the variance of four of the elementary arithmetic operations are

$$\sigma_{m_1+m_2}^{2} = \sigma_{m_1}^{2} + \sigma_{m_2}^{2} \qquad \text{(A.4)}$$

$$\sigma_{m_1-m_2}^{2} = \sigma_{m_1}^{2} + \sigma_{m_2}^{2} \qquad \text{(A.5)}$$

$$\frac{\sigma_{m_1m_2}^{2}}{m_1^{2}m_2^{2}} = \frac{\sigma_{m_1}^{2}}{m_1^{2}} + \frac{\sigma_{m_2}^{2}}{m_2^{2}} \qquad \text{(A.6)}$$

$$\frac{\sigma_{m_1/m_2}^{2}}{m_1^{2}/m_2^{2}} = \frac{\sigma_{m_1}^{2}}{m_1^{2}} + \frac{\sigma_{m_2}^{2}}{m_2^{2}} \qquad \text{(A.7)}$$

where m_1 and m_2 are two independent counting rates.

Example 2: If $I_d = 100 \pm 3.2$ cpm

$I_w = 40 \pm 1.6$ cpm

then

$$I_d - I_w = 100 - 40 = 60 \text{ cpm} \quad ,$$

its standard deviation is

$$\sigma_{I_d-I_w} = \left(1.6^2 + 3.2^2\right)^{\frac{1}{2}} = 3.6$$

and

$$I_d - I_w = 60 \pm 3.6 \text{ cpm} \quad .$$

Example 3:

When $I_d = 100 \pm 40$ cpm

$I_w = 10 \pm 3$ cpm

then

$$\frac{I_d}{I_w} = \frac{100 \pm 40}{10 \pm 3} = 10 \pm 10\left[(40/100)^2 + (3/10)^2\right]^{\frac{1}{2}}$$
$$= 10 \pm 5 \text{ cpm}$$

The effect of fluctuations in the counting rate due to the random disintegrations of the γ-ray source will produce a corresponding effect on the moisture content calculation. The attenuation equation used to compute the moisture content is

$$\frac{I_w}{I_d} = e^{-\sigma_w x_w} \qquad (7.4)$$

Multiplying and dividing by ℓ/ρ in the exponent of Eq. (7.4) yields

$$\frac{I_w}{I_d} = e^{-(\sigma_w \ell/\rho)(\rho x_w/\ell)} \qquad (A.8)$$

where ℓ (cm) is the distance between the walls of the plastic tube and ρ is the density of water in gm/cm^3. The quantity $\rho x_w/\ell$ is the moisture content θ in gm/cm^3 or cm^3/cm^3. Equation (A.8) may be written as

$$\frac{I_w}{I_d} = e^{-(\sigma_w \ell/\rho)\theta}$$

Solving for θ yields

$$\theta = -\frac{\rho}{\sigma_w \ell} \ln \frac{I_w}{I_d}$$

Differentiating with respect to I_w/I_d ,

$$\frac{d\theta}{d\left(\frac{I_w}{I_d}\right)} = - \frac{\rho}{\sigma_w \ell} \frac{I_d}{I_w} \tag{A.9}$$

For small changes, Eq. (A.9) approximates

$$\Delta(\theta) = - \frac{\rho}{\sigma_w \ell} \frac{I_d}{I_w} \Delta\left(\frac{I_w}{I_d}\right) \tag{A.10}$$

For a change of I_w/I_d equal to σ_{I_w/I_d} , Eq. (A.10) becomes

$$\Delta(\theta) = - \frac{\rho}{\sigma_w \ell} \frac{I_d}{I_w} \frac{I_w}{I_d} \left[\left(\sigma_w/I_w\right)^2 + \left(\sigma_d/I_d\right)^2\right]^{\frac{1}{2}}$$

or

$$\Delta(\theta) = - \frac{\rho}{\sigma_w \ell} \left[\left(\sigma_w/I_w\right)^2 + \left(\sigma_d/I_d\right)^2\right]^{\frac{1}{2}} \tag{A.11}$$

where σ_w is the standard deviation of I_w and σ_d is the standard deviation of I_d . Substituting $\sigma_w = (I_w)^{\frac{1}{2}}$ and $\sigma_d = (I_d)^{\frac{1}{2}}$ gives

$$\Delta(\theta) = - \frac{\rho}{\sigma_w \ell} \left(\frac{I_w + I_d}{I_w \cdot I_d}\right)^{\frac{1}{2}} \tag{A.12}$$

The fractional variation is

$$\frac{\Delta(\theta)}{\theta} = \frac{- \frac{\rho}{\sigma_w \ell} \left(\frac{I_w + I_d}{I_w \cdot I_d}\right)^{\frac{1}{2}}}{- \frac{\rho}{\sigma_w \ell} \ln \frac{I_w}{I_d}} = \frac{\pm \left(\frac{I_w + I_d}{I_w \cdot I_d}\right)^{\frac{1}{2}}}{\ln \frac{I_w}{I_d}} \tag{A.13}$$

In the experimental program a typical value for I_d was 36,000 cpm and I_w was about 28,000 cpm when the soil became saturated. Table A.1 shows the fractional and total variations in the moisture content determinations.

TABLE A.1

FRACTIONAL AND TOTAL VARIATIONS IN THE MOISTURE CONTENT DETERMINATIONS

I_d	I_w	I_w/I_d	θ	$\Delta(\theta)$	$\Delta(\theta)/\theta$
36,000	35,000	0.972	0.0538	0.01420	0.264
36,000	34,000	0.945	0.1072	0.01428	0.133
36,000	33,000	0.916	0.1662	0.01442	0.0868
36,000	32,000	0.889	0.2220	0.01450	0.0653
36,000	31,000	0.861	0.2830	0.01462	0.0518
36,000	30,000	0.833	0.3460	0.01480	0.0428
36,000	29,000	0.805	0.4090	0.01491	0.0365
36,000	28,000	0.778	0.4760	0.01511	0.0318

Method of Reducing the Errors

1. Increase the counting period as shown in Eq. (A.3).
2. Increase the counting rate. This effect will enter through the term $\left[(I_w + I_d)/(I_w \cdot I_d)\right]^{\frac{1}{2}}$ in Eq. (A.13). The counting rate may be increased by using a stronger source or by employing a sample of smaller cross-section. By reference to the denominator of Eq. (A.13) it is seen that greater relative accuracy would result if the specific gravity of the soil were reduced. Hence the use of a medium having a specific gravity less than 2.65 would be advantageous in experimental work where gamma-attenuation techniques are employed.

A.4 Computer Program for Statistical Analysis

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                    STANFORD B5500 ALGOL -- 12/10/65 VERSION        060/66
BEGIN COMMENT CALCULATION OF INFILTRATION RATE OF VERTICAL FLOW
WITH CERTAIN APPLIED PRESSURE AND ASSUMING COMPLETE SATURATION UP
TO THE WETTING FRONT
P COMPRISES THE APPLIED PRESSURE AND THE CAPILLARY FORCES;
REAL    P,K,N,X,V,X1,X2,X3,Y1,Y2,Y3;
INTEGER T;
LABEL L1,L2,L3;
FORMAT FMT1(X10,"P",X19,"K",X14,"N");
FORMAT FMT2(X10,"T",X15,"X",X15,"V");
L1: READ(P,K,N);
WRITE(FMT1);
WRITE(<X5,3(F12.5,X5)>,P,K,N);
WRITE([DBL]);
WRITE(FMT2);
FOR T←5 STEP 5 UNTIL 100 DO
    BEGIN
X1←0;  X2←200;
L3: Y1←(N/K)×(X1+P×(LN(P/( X1+P))))-T;
    Y2←(N/K)×(X2+P×(LN(P/( X2+P))))-T;
    IF ABS(X1-X2)<0.1 THEN
         BEGIN X←X2; GO TO L2; END;
    X3←0.5×(X1+X2);
    Y3←(N/K)×(X3+P×(LN(P/( X3+P))))-T;
IF Y3=0 THEN
         BEGIN X←X3; GO TO L2; END;
    IF (Y1×Y3)<0 THEN
         BEGIN X2←X3; GO TO L3;  END;
    X1←X3; GO TO L3;
L2: V←K×((P+X)/X);
WRITE(<X5,3(F12.5,X4)>,T,X,V);
    END;
WRITE([PAGE]);
GO TO L1;
END.
```

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